APPENDICES

Appendix A Derivation of the Infinitesimal Generator

For completeness, we recall the multidimensional Itô formula (see Oksendal, 2013, and Fleming & Soner, 2006). Let

$$X(t) = \big(X_1(t), X_2(t), \ldots, X_n(t)\big)$$

be an (n)-dimensional diffusion process satisfying

$$dX_i(t) = \mu_i(t, X), dt + \sum_{j=1}^{m} \sigma_{ij}(t, X), dW_j(t), \qquad i = 1, \ldots, n, \tag{Appendix A.1}$$

where $(W(t) = (W_1(t), \ldots, W_m(t))^\top)$ is an (m)-dimensional Brownian motion. If $(G(t, X) \in C^{1,2}([0, T] \times \mathbb{R}^n))$, then Itô's formula yields

$$dG(t, X(t)) =, G_t(t, X), dt + \sum_{i=1}^{n} G_{x_i}(t, X), dX_i(t) \quad + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} G_{x_i x_j}(t, X), dX_i(t), dX_j(t). \tag{Appendix A.2}$$

Substituting (Appendix A.1) into (Appendix A.2) gives

$$dG(t, X(t)) = \left[G_t + \sum_{i=1}^{n} \mu_i G_{x_i} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} G_{x_i x_j} \right] dt \quad + \sum_{k=1}^{m} \left(\sum_{i=1}^{n} \sigma_{ik} G_{x_i} \right) dW_k(t), \tag{Appendix A.3}$$

where

$$a_{ij} = \sum_{k=1}^{m} \sigma_{ik} \sigma_{jk}, \qquad i, j = 1, \ldots, n, \tag{Appendix A.4}$$

is the covariance matrix of the diffusion process.

For our problem by applying Itô's formula to the value function $(G(t, x_1, x_2, s_1, s_2))$, we obtain

$$
\begin{aligned}
dG =&, G_t, dt + G_{x_1}, dX_1 + G_{x_2}, dX_2 + G_{s_1}, dS_1 + G_{s_2}, dS_2 \\
&+ \frac{1}{2}G_{x_1x_1}(dX_1)^2 + \frac{1}{2}G_{x_2x_2}(dX_2)^2 + \frac{1}{2}G_{s_1s_1}(dS_1)^2 + \frac{1}{2}G_{s_2s_2}(dS_2)^2 \\
&+ G_{x_1x_2}dX_1dX_2 + G_{x_1s_1}dX_1dS_1 + G_{x_2s_2}dX_2dS_2.
\end{aligned}
\tag{Appendix A.5}
$$

Substituting the controlled wealth dynamics for insurer and reinsurer we have the following:

$$
\begin{aligned}
dX_1(t) =& \Big[r_0X_1 + \pi_1(t)\Big(\phi_1(\alpha_1 + (1-\lambda_1)\delta_1 - \ln S_1) - r_0\Big) + a(\eta_1 - \eta_3 q(t))\Big]dt \\
&+ \pi_1(t)\sigma_1, dW_1 + b_1(1-q(t)), dW_0,
\end{aligned}
\tag{Appendix A.6}
$$

$$
\begin{aligned}
dX_2(t) =& \Big[r_0X_2 + \pi_2(t)\Big(\phi_2(\alpha_2 + (1-\lambda_2)\delta_2 - \ln S_2) - r_0\Big) + a\eta_3 q(t)\Big]dt \\
&+ \pi_2(t)\sigma_2, dW_2 + b_1 q(t), dW_0,
\end{aligned}
\tag{Appendix A.7}
$$

together with the risky-asset dynamics for the insurer and reinsurer we have the following:

$$
dS_1(t) = \phi_1\Big(\alpha_1 + (1-\lambda_1)\delta_1 - \ln S_1\Big)S_1, dt + \sigma_1 S_1, dW_1, \tag{Appendix A.8}
$$

and

$$
dS_2(t) = \phi_2\Big(\alpha_2 + (1-\lambda_2)\delta_2 - \ln S_2\Big)S_2, dt + \sigma_2 S_2, dW_2, \tag{Appendix A.9}
$$

into (Appendix A.5) yields

$$
\begin{aligned}
dG = {} & G_t\, dt \\
& + G_{x_1}\Big\{\Big[r_0 X_1 + \pi_1\Big(\phi_1(\alpha_1 + (1-\lambda_1)\delta_1 - \ln S_1) - r_0\Big) + a(\eta_1 - \eta_3 q)\Big]dt \\
& \qquad + \pi_1\sigma_1\, dW_1 + b_1(1-q)\, dW_0\Big\} \\
& + G_{x_2}\Big\{\Big[r_0 X_2 + \pi_2\Big(\phi_2(\alpha_2 + (1-\lambda_2)\delta_2 - \ln S_2) - r_0\Big) + a\eta_3 q\Big]dt \\
& \qquad + \pi_2\sigma_2\, dW_2 + b_1 q\, dW_0\Big\} \\
& + G_{s_1}\Big\{\phi_1\Big(\alpha_1 + (1-\lambda_1)\delta_1 - \ln S_1\Big)S_1\, dt + \sigma_1 S_1\, dW_1\Big\} \\
& + G_{s_2}\Big\{\phi_2\Big(\alpha_2 + (1-\lambda_2)\delta_2 - \ln S_2\Big)S_2\, dt + \sigma_2 S_2\, dW_2\Big\} \\
& + \frac{1}{2}G_{x_1x_1}(dX_1)^2 + \frac{1}{2}G_{x_2x_2}(dX_2)^2 + \frac{1}{2}G_{s_1s_1}(dS_1)^2 + \frac{1}{2}G_{s_2s_2}(dS_2)^2 \\
& + G_{x_1x_2}dX_1dX_2 + G_{x_1s_1}dX_1dS_1 + G_{x_2s_2}dX_2dS_2.
\end{aligned}
\tag{Appendix A.10}
$$

Using the Itô multiplication rules

$$(dW_i)^2 = dt, \qquad dW_i dW_j = 0, \quad i \neq j,$$

we obtain

$$(dX_1)^2 = \Big[\pi_1^2\sigma_1^2 + b_1^2(1-q)^2\Big]dt, \tag{Appendix A.11}$$

$$(dX_2)^2 = \Big[\pi_2^2\sigma_2^2 + b_1^2 q^2\Big]dt, \tag{Appendix A.12}$$

$$(dS_1)^2 = \sigma_1^2 S_1^2 dt, \tag{Appendix A.13}$$

$$(dS_2)^2 = \sigma_2^2 S_2^2 dt, \tag{Appendix A.14}$$

The mixed quadratic variation term dX_1dX_2 arises because both wealth processes are driven by the common Brownian motion $W^0(t)$, which represents the aggregate insurance risk.

$$dX_1dX_2 = b_1^2 q(1-q), dt, \tag{Appendix A.15}$$

The mixed quadratic variation term dX_1dS_1 arises because both the insurer's wealth process $X_1(t)$ and the risky asset price process $S_1(t)$ are driven by the common Brownian motion $W^1(t)$. This common source of uncertainty generates the cross-partial derivative term $G_{x_1s_1}$ in the infinitesimal generator.

$$dX_1dS_1 = \pi_1\sigma_1^2 S_1, dt, \tag{Appendix A.16}$$

The mixed quadratic variation term $dX_2 dS_2$ arises because both the reinsurer's wealth process $X_2(t)$ and the risky asset price process $S_2(t)$ are driven by the common Brownian motion $W^2(t)$. This common source of uncertainty generates the cross-partial derivative term $G_{x_2 s_2}$ in the infinitesimal generator.

$$dX_2 dS_2 = \pi_2 \sigma_2^2 S_2, dt. \tag{Appendix A.17}$$

Substituting the above quadratic variation terms into the drift component of (Appendix A.5) and collecting like terms yields

$$dG = \mathcal{A}^{\kappa_1,\kappa_2} G\, dt + \text{martingale terms.}$$

Hence, the infinitesimal generator is given by

$$\begin{aligned}\mathcal{A}^{\kappa_1,\kappa_2} G =& G_t + \Big[r_0 x_1 + \pi_1(t)\Big(\phi_1\big(\alpha_1 + (1-\lambda_1)\delta_1 - \ln s_1\big) - r_0\Big) + a\big(\eta_1 - \eta_3 q(t)\big)\Big] G_{x_1} \\ & + \Big[r_0 x_2 + \pi_2(t)\Big(\phi_2\big(\alpha_2 + (1-\lambda_2)\delta_2 - \ln s_2\big) - r_0\Big) + a\eta_3 q(t)\Big] G_{x_2} \\ & + \Big[\phi_1\big(\alpha_1 + (1-\lambda_1)\delta_1 - \ln s_1\big)s_1\Big] G_{s_1} + \Big[\phi_2\big(\alpha_2 + (1-\lambda_2)\delta_2 - \ln s_2\big)s_2\Big] G_{s_2} \\ & + \frac{1}{2}\Big[\pi_1^2(t)\sigma_1^2 + b_1^2(1-q(t))^2\Big] G_{x_1x_1} + \frac{1}{2}\Big[\pi_2^2(t)\sigma_2^2 + b_1^2 q^2(t)\Big] G_{x_2x_2} \\ & + \frac{1}{2}\sigma_1^2 s_1^2 G_{s_1s_1} + \frac{1}{2}\sigma_2^2 s_2^2 G_{s_2s_2} + \pi_1(t)\sigma_1^2 s_1, G_{x_1s_1} + \pi_2(t)\sigma_2^2 s_2, G_{x_2s_2} + \\ & b_1^2 q(t)\big(1-q(t)\big) G_{x_1x_2}.\end{aligned} \tag{Appendix A.18}$$

which corresponds to Equation (2.14) in the main text.

The application of Itô's formula produces both drift terms (proportional to dt) and stochastic terms (proportional to $dW_0(t)$, $dW_1(t)$, and $dW_2(t)$). The stochastic integrals with respect to these Brownian motions define local martingales. Since

$$\mathbb{E}_t[dW_i(t)] = 0, \qquad i = 0, 1, 2,$$

their conditional expectations vanish. Consequently, the stochastic terms do not contribute to the infinitesimal generator. More precisely, the generator is defined by

$$\mathcal{A}^{\kappa_1,\kappa_2} G = \lim_{\Delta t \to 0} \frac{\mathbb{E}_t\Big[G(t+\Delta t, X(t+\Delta t)) - G(t, X(t))\Big]}{\Delta t},$$

where $\mathbb{E}_t[\cdot]$ denotes the conditional expectation given the information available at time t. Since the martingale component has zero conditional expectation, only the drift component contributes to the above limit. Therefore, the infinitesimal generator $\mathcal{A}^{\kappa_1,\kappa_2}$ is obtained by collecting all terms multiplied by dt in the Itô expansion, while the stochastic dW-terms vanish after taking conditional expectations.

Appendix B Proof of Theorem 1

Proof. Let $(\pi_1, \pi_2, q) \in \mathcal{A}$ be an arbitrary admissible control, and let

$$(X_1, X_2, S_1, S_2)$$

denote the corresponding controlled state process. Since the coefficients of the state equations are continuous and satisfy the standard local Lipschitz and linear growth conditions, the state process admits a unique strong solution.

Moreover, the candidate value function

$$V \in C^{1,2,2,2,2}$$

satisfies the growth condition stated in Assumption 2. Together with the admissibility condition

$$\mathbb{E} \int_0^T \left(|\pi_1(t)|^2 + |\pi_2(t)|^2 + |q(t)|^2 \right) dt < \infty,$$

this implies

$$\mathbb{E} \left[\sup_{0 \le t \le T} |V(t, X_1(t), X_2(t), S_1(t), S_2(t))| \right] < \infty,$$

so that all expectations appearing below are well defined.
Applying Itô's formula to

$$V(t, X_1(t), X_2(t), S_1(t), S_2(t))$$

gives

$$dV = \Big(V_t + \mathcal{L}^{\pi_1, \pi_2, q} V \Big) dt + dM_t,$$

where (M_t) is a local martingale. Since the admissible controls satisfy the square-integrability condition and the candidate value function satisfies the growth condition in Assumption 2, the stochastic integral defining (M_t) is square integrable. Consequently, by standard martingale theory, (M_t) is a true martingale and therefore

$$\mathbb{E}[M_T - M_t \mid \mathcal{F}_t] = 0.$$

Integrating from (t) to (T) yields

$$V(T, X_1(T), X_2(T), S_1(T), S_2(T)) = V(t, x_1, x_2, s_1, s_2) + \\ \int_t^T \Big(V_u + \mathcal{L}^{\pi_1, \pi_2, q} V \Big) du + M_T - M_t.$$

Since (V) satisfies the HJB equation,

$$\sup_{(\pi_1,\pi_2,q)\in\mathcal{A}}\Big\{V_t+\mathcal{L}^{\pi_1,\pi_2,q}V\Big\}=0, \qquad \text{(Appendix B.1)}$$

it follows that

$$V_t+\mathcal{L}^{\pi_1,\pi_2,q}V\le 0.$$

Hence

$$V(T,X_1(T),X_2(T),S_1(T),S_2(T))\le V(t,x_1,x_2,s_1,s_2)+M_T-M_t.$$

Taking conditional expectations and using the martingale property gives

$$\mathbb{E}\Big[V(T,X_1(T),X_2(T),S_1(T),S_2(T))\mid\mathcal{F}_t\Big]\le V(t,x_1,x_2,s_1,s_2).$$

Using the terminal condition,

$$\mathbb{E}\Big[U(X_1(T),X_2(T))\mid\mathcal{F}_t\Big]\le V(t,x_1,x_2,s_1,s_2).$$

Since the admissible control was arbitrary,

$$G(t,x_1,x_2,s_1,s_2)\le V(t,x_1,x_2,s_1,s_2).$$

Now consider the admissible control

$$(\pi_1^*,\pi_2^*,q^*).$$

Since the admissible control

$$(\pi_1^*,\pi_2^*,q^*)$$

attains the supremum in the HJB equation,

$$V_t+\mathcal{L}^{\pi_1^*,\pi_2^*,q^*}V=0.$$

Furthermore, by Assumption 5, the control

$$(\pi_1^*,\pi_2^*,q^*)$$

is admissible and therefore belongs to the admissible control set $\mathcal{A}$. Consequently, the above argument applies equally to the candidate optimal control. Applying Itô's formula again yields

$$V(T,X_1^*(T),X_2^*(T),S_1^*(T),S_2^*(T))=V(t,x_1,x_2,s_1,s_2)+M_T-M_t.$$

Since all expectations are finite by Assumption 2 and the martingale property established above, each expectation below is well defined.

Taking conditional expectations gives

$$V(t,x_1,x_2,s_1,s_2) = \mathbb{E}\Big[U(X_1^*(T), X_2^*(T)) \mid \mathcal{F}_t\Big].$$

Therefore

$$V(t,x_1,x_2,s_1,s_2) \leq G(t,x_1,x_2,s_1,s_2).$$

Combining

$$G(t,x_1,x_2,s_1,s_2) \leq V(t,x_1,x_2,s_1,s_2)$$

and

$$V(t,x_1,x_2,s_1,s_2) \leq G(t,x_1,x_2,s_1,s_2)$$

yields

$$V(t,x_1,x_2,s_1,s_2) = G(t,x_1,x_2,s_1,s_2).$$

Hence, (π_1^*, π_2^*, q^*) is optimal.

$\square$

Corollary 1. *Let $V(t,x_1,x_2,s_1,s_2)$ be the candidate value function obtained and let (π_1^*, π_2^*, q^*) denote the corresponding candidate optimal controls derived from the first-order conditions of the HJB equation. If (V) satisfies the assumptions of the Verification Theorem and the controls (π_1^*, π_2^*, q^*) are admissible, then*

$$V(t,x_1,x_2,s_1,s_2) = G(t,x_1,x_2,s_1,s_2),$$

and the controls (π_1^*, π_2^*, q^*) are optimal.

Appendix C Verification of Optimality Conditions

To verify that the candidate controls correspond to optimal solutions, we examine the second-order derivatives of the Hamiltonian with respect to the control variables.

For the investment controls, we obtain

$$\frac{\partial^2}{\partial \pi_1^2}\left(\frac{\mathcal{A}^{\kappa_1,\kappa_2}G}{G}\right) = -\beta_1\sigma_1^2 e^{r_0(T-t)},$$

and

$$\frac{\partial^2}{\partial \pi_2^2}\left(\frac{\mathcal{A}^{\kappa_1,\kappa_2}G}{G}\right) = -\beta_2\sigma_2^2 e^{r_0(T-t)}.$$

Since

$$\beta_1 > 0, \qquad \beta_2 > 0, \qquad \sigma_1^2 > 0, \qquad \sigma_2^2 > 0,$$

it follows that

$$\frac{\partial^2}{\partial \pi_1^2}\left(\frac{\mathcal{A}^{\kappa_1,\kappa_2}G}{G}\right) < 0, \qquad \frac{\partial^2}{\partial \pi_2^2}\left(\frac{\mathcal{A}^{\kappa_1,\kappa_2}G}{G}\right) < 0.$$

Hence, the Hamiltonian is concave with respect to the investment controls $\pi_1(t)$ and $\pi_2(t)$.

For the reinsurance control $q(t)$, differentiating the $q(t)$-dependent part of the infinitesimal generator $\mathcal{A}^{\kappa_1,\kappa_2}G$ with respect to $q(t)$ gives

$$\frac{\partial}{\partial q(t)}\left(\mathcal{A}^{\kappa_1,\kappa_2}G\right) = -\eta_3 a G_{x_1} + \eta_3 a G_{x_2} - b_1^2(1-q)G_{x_1x_1} + b_1^2 q G_{x_2x_2} + b_1^2(1-2q)G_{x_1x_2}.$$

Setting

$$\frac{\partial}{\partial q(t)}\left(\mathcal{A}^{\kappa_1,\kappa_2}G\right) = 0$$

yields the candidate optimal reinsurance strategy $q(t)$.
Differentiating once more with respect to $q(t)$ gives

$$\frac{\partial^2}{\partial q(t)^2}\left(\mathcal{A}^{\kappa_1,\kappa_2}G\right) = b_1^2 G_{x_1x_1} + b_1^2 G_{x_2x_2} - 2b_1^2 G_{x_1x_2}.$$

Substituting

$$G_{x_1x_1} = \beta_1^2 e^{2r_0(T-t)}G, \qquad G_{x_2x_2} = \beta_2^2 e^{2r_0(T-t)}G, \qquad G_{x_1x_2} = \beta_1\beta_2 e^{2r_0(T-t)}G,$$

yields

$$\frac{\partial^2}{\partial q(t)^2}\left(\mathcal{A}^{\kappa_1,\kappa_2}G\right) = b_1^2 e^{2r_0(T-t)}G\left(\beta_1^2 + \beta_2^2 - 2\beta_1\beta_2\right).$$

Hence,

$$\frac{\partial^2}{\partial q(t)^2}\left(\mathcal{A}^{\kappa_1,\kappa_2}G\right) = b_1^2(\beta_1-\beta_2)^2 e^{2r_0(T-t)}G.$$

The sign of this expression depends on the sign convention adopted for the value function. Consequently, the admissibility conditions and the multiple-case analysis presented in Theorem 2 are used to determine whether the interior critical point or one of the boundary controls yields the optimal reinsurance strategy. Accordingly, the optimal solution may correspond to the interior dynamic-retention regime or to one of the boundary cases $q^*(t) = 0$ or $q^*(t) = 1$, depending on the parameter configuration and the admissibility constraints. For the investment controls $\pi_1(t)$ and $\pi_2(t)$, the second-order derivatives remain negative, implying concavity of the Hamiltonian with respect to these controls. Therefore, the first-order conditions yield the optimal investment strategies $\pi_1^*(t)$ and $\pi_2^*(t)$.

Appendix D Table of results and Proof of Theorem 2

Table 1: Classification of the optimal reinsurance regimes

Optimal retention $q^*(t)$	**Conditions**	**Case**
$q^*(t) = \frac{\eta_1}{\eta_3} \quad t_1 \le t \le T$ (Fixed proportional risk sharing. The retention level remains constant and is independent of market conditions.)	$1 < \frac{a\eta_3^2}{b_1^2\beta_2\eta_1 + b_1^2\beta_1(\eta_3 - \eta_1)}$ and $\frac{a\eta_3}{b_1^2\beta_1} < e^{r_0 T}$ Equation (Appendix D.14) provides the value function.	(i)
	$1 < \frac{a\eta_3^2}{b_1^2\beta_2\eta_1 + b_1^2\beta_1(\eta_3 - \eta_1)}$, $\frac{a\eta_3^2}{b_1^2\beta_2\eta_1 + b_1^2\beta_1(\eta_3 - \eta_1)} \le e^{r_0 t} \le \frac{a\eta_3}{b_1^2\beta_1}$, $(\pi_1^*(t), \pi_2^*(t))$ are $t_1 \le t \le T$. shown in 4.1 where (Appendix D.33) provides the value function.	(iv)
	$1 < \frac{a\eta_3^2}{b_1^2\beta_2\eta_1 + b_1^2\beta_1(\eta_3 - \eta_1)}$, $\frac{a\eta_3^2}{b_1^2\beta_2\eta_1 + b_1^2\beta_1(\eta_3 - \eta_1)} > e^{r_0 t}, \quad 0 \le t \le T.$ $(\pi_1^*(t), \pi_2^*(t))$ are shown in 4.1 where (Appendix D.37) provides the value function.	(vi)
$q^*(t) = 0$ (Full risk transfer to the reinsurer. Retaining risk is not economically optimal.)	$1 < \frac{a\eta_3^2}{b_1^2\beta_2\eta_1 + b_1^2\beta_1(\eta_3 - \eta_1)}, \frac{a\eta_3}{b_1^2\beta_1} < e^{r_0 T},$ $(\pi_1^*(t), \pi_2^*(t))$ $0 \le t \le t_0$. are shown in 4.1 where (Appendix D.30) provides the value function.	(iii)
	$0 < \frac{a\eta_3^2}{b_1^2\beta_2\eta_1 + b_1^2\beta_1(\eta_3 - \eta_1)} \le 1,$ $1 < \frac{a\eta_3}{b_1^2\beta_1} < e^{r_0 T}, \frac{a\eta_3}{b_1^2\beta_1} < e^{r_0 T}, \quad 0 \le t \le t_0.$ $(\pi_1^*(t), \pi_2^*(t))$ are shown in 4.1 where (Appendix D.41) provides the value function.	(viii)

Continued on next page

<table>
<tr><th>Optimal retention $q^*(t)$</th><th>Conditions</th><th>Case</th></tr>
<tr><td></td><td>$0 < \frac{a\eta_3^2}{b_1^2\beta_2\eta_1 + b_1^2\beta_1(\eta_3 - \eta_1)} \le 1,$
$0 < \frac{a\eta_3}{b_1^2\beta_1} \le 1, \quad 0 \le t \le T.$ $(\pi_1^*(t), \pi_2^*(t))$ are shown in 4.1 where (Appendix D.45) provides the value function.</td><td>(x)</td></tr>
<tr><td rowspan="4">$q^*(t) =$
$\frac{\beta_1}{\beta_1 - \beta_2} + \frac{\eta_3 a}{(\beta_1 - \beta_2)b_1^2 e^{r_0(T-t)}}$
(Dynamic risk sharing. Retention varies with risk preferences and time to maturity.)</td><td>$1 < \frac{a\eta_3^2}{b_1^2\beta_2\eta_1 + b_1^2\beta_1(\eta_3 - \eta_1)}, \frac{a\eta_3}{b_1^2\beta_1} < e^{r_0 T},$ $(\pi_1^*(t), \pi_2^*(t))$
$t_0 \le t \le t_1.$
are shown in 4.1 where (Appendix D.23) provides the value function.</td><td>(ii)</td></tr>
<tr><td>$1 < \frac{a\eta_3^2}{b_1^2\beta_2\eta_1 + b_1^2\beta_1(\eta_3 - \eta_1)},$
$\frac{a\eta_3^2}{b_1^2\beta_2\eta_1 + b_1^2\beta_1(\eta_3 - \eta_1)} \le e^{rt} \le \frac{a\eta_3}{b_1^2\beta_1},$ $(\pi_1^*(t), \pi_2^*(t))$ are
$0 \le t \le t_1.$
shown in 4.1 where (Appendix D.35) provides the value function.</td><td>(v)</td></tr>
<tr><td>$0 < \frac{a\eta_3^2}{b_1^2\beta_2\eta_1 + b_1^2\beta_1(\eta_3 - \eta_1)} \le 1,$
$1 < \frac{a\eta_3}{b_1^2\beta_1} < e^{r_0 T}, \frac{a\eta_3}{b_1^2\beta_1} < e^{r_0 T}, \quad t_0 < t \le T.$
$(\pi_1^*(t), \pi_2^*(t))$ are shown in 4.1 where (Appendix D.39) provides the value function.</td><td>(vii)</td></tr>
<tr><td>$0 < \frac{a\eta_3^2}{b_1^2\beta_2\eta_1 + b_1^2\beta_1(\eta_3 - \eta_1)} \le 1,$ $(\pi_1^*(t), \pi_2^*(t))$
$1 < \frac{a\eta_3}{b_1^2\beta_1} < e^{r_0 T}, \frac{a\eta_3}{b_1^2\beta_1} \ge e^{rt}, \quad 0 \le t \le T.$
are shown in 4.1 where (Appendix D.43) provides the value function.</td><td>(ix)</td></tr>
</table>

The proof of Theorem 2 is obtained by applying the dynamic programming principle and solving the associated Hamilton–Jacobi–Bellman (HJB) equations. By substituting the candidate value function into the HJB system and deriving the first-order optimality conditions, the explicit forms of the optimal reinsurance and investment strategies are obtained. Furthermore, the admissibility conditions and parameter restrictions divide the problem into several cases, each corresponding to different regions of the control space. The corresponding value functions are then verified by substituting the optimal controls back into the HJB equations, which completes the proof.

Case 1. $1 < \frac{a\eta_3^2}{b_1^2\beta_2\eta_1 + b_1^2\beta_1(\eta_3 - \eta_1)}$ *and* $\frac{a\eta_3}{b_1^2\beta_1} < e^{r_0 T}$.

Equation (3.8) shows that $\bar{q}(t) \in \left(0, \frac{\eta_1}{\eta_3}\right)$ is equivalent to

$$t \leq t_1 = T - \frac{1}{r_0} \ln \frac{a\eta_3^2}{b_1^2\beta_2\eta_1 + b_1^2\beta_1(\eta_3 - \eta_1)}.$$

and

$$t \geq t_0 = T - \frac{1}{r_0} \ln \frac{a\eta_3}{b_1^2\beta_1}.$$

(a.) For $t_1 \leq t \leq T$ we have $\bar{q} \geq \frac{\eta_1}{\eta_3}$.
Then;

$$q^*(t) = \frac{\eta_1}{\eta_3} \qquad t_1 \leq t \leq T. \tag{Appendix D.1}$$

Substituting (3.5), (3.6), and (Appendix D.1) into the HJB equation (2.14), and dividing by the strictly positive function $G(t, x_1, x_2, s_1, s_2)$, yields a reduced nonlinear equation. By collecting terms involving only t, only (t, s_1), and only (t, s_2), the equation can be separated into independent components. Hence, the following system of equations is obtained:

$$\begin{aligned} &h_t - r_0 h(t) + \eta_1 a\beta_2 - \Bigg[b_1^2\beta_1^2 + \frac{(\beta_1 - \frac{1}{2}\beta_2)(-2\eta_1 b_1^2\beta_1)}{\eta_3} \\ &+ \frac{(\beta_1 - \frac{1}{2}\beta_2)(2\eta_1^2 b_1^2\beta_1)}{\eta_3^2}\Bigg] e^{r_0(T-t)} = 0. \end{aligned} \tag{Appendix D.2}$$

$$\begin{aligned} &f_t + r_0 s_1(t) f_{s_1} + \frac{1}{2}\sigma_1^2 s_1^2(t) f_{s_1 s_1} \\ &- \frac{1}{2}\frac{\left[(\phi_1\big(\alpha_1 + (1-\lambda_1)\delta_1 - \ln s_1(t)\big) - r_0\right]^2}{\sigma_1^2} = 0. \end{aligned} \tag{Appendix D.3}$$

$$\begin{aligned} &g_t + r_0 s_2(t) g_{s_2} + \frac{1}{2}\sigma_2^2 s_2^2(t) g_{s_2 s_2} \\ &- \frac{1}{2}\frac{\left[(\phi_2\big(\alpha_2 + (1-\lambda_2)\delta_2 - \ln s_2(t)\big) - r_0\right]^2}{\sigma_2^2} = 0. \end{aligned} \tag{Appendix D.4}$$

When $h(T) = 0$, the solution to (Appendix D.2) is as follows;

$$\begin{aligned} &h(t) = \frac{\eta_1 a\beta_2}{r_0}(1 - e^{r_0(T-t)}) - \\ &\frac{1}{2r_0}\left[b_1^2\beta_1^2 + \frac{(\beta_1 - \frac{1}{2}\beta_2)(-2\eta_1 b_1^2\beta_1)}{\eta_3} + \frac{(\beta_1 - \frac{1}{2}\beta_2)(2\eta_1^2 b_1^2\beta_1)}{\eta_3^2}\right] \\ &(e^{r_0(T-t)} - e^{-r_0(T-t)}). \end{aligned} \tag{Appendix D.5}$$

Equations (Appendix D.3) and (Appendix D.4) will be solved using the variable change technique and power transformation to become linear equations.

For equation (Appendix D.3), we are going to have the following power transformation.

$$w = s_1^{-2}, \qquad f(t, s_1) = F(t, w).$$

Applying the chain rule gives

$$f_t = F_t,$$

$$f_{s_1} = \frac{\partial F}{\partial w}\frac{\partial w}{\partial s_1} = -2s_1^{-3}F_w,$$

and

$$f_{s_1 s_1} = 6s_1^{-4}F_w + 4s_1^{-6}F_{ww}.$$

with the boundary condition $F(T, w) = 0$.
Substituting these expressions into equation (Appendix D.3) yields the transformed equation.

$$\begin{aligned} &F_t - 2r_0 wF_w + 3\sigma_1^2 wF_w + 2\sigma_1^2 w^2 F_{ww} \\ &- \frac{1}{2}\frac{\left[(\phi_1\big(\alpha_1 + (1-\lambda_1)\delta_1 - \ln s_1(t)\big) - r_0\right]^2}{\sigma_1^2}. = 0. \end{aligned} \tag{Appendix D.6}$$

In order to solve (Appendix D.6), we have to do the following:

$$F(t, w) = J(t) + K(t)w. \tag{Appendix D.7}$$

where $J(T) = K(T) = 0$.
By substituting the equation (Appendix D.7) and its derivatives into (Appendix D.6) and splitting into the following two equations:

$$J_t - \frac{1}{2}\frac{\left[(\phi_1\big(\alpha_1 + (1-\lambda_1)\delta_1 - \ln s_1(t)\big) - r_0\right]^2}{\sigma_1^2} = 0. \tag{Appendix D.8}$$

$$K_t - 2r_0 wK(t) + 3\sigma_1^2 wK(t) = 0. \tag{Appendix D.9}$$

We obtain the solutions to (Appendix D.8) and (Appendix D.9) by taking the boundary conditions $J(T) = K(T) = 0$.

$$J(t) = \frac{1}{2}\frac{\left[(\phi_1\big(\alpha_1 + (1-\lambda_1)\delta_1 - \ln s_1(t)\big) - r_0\right]^2}{\sigma_1^2}(T - t). \tag{Appendix D.10}$$

$$K(t) = 0. \tag{Appendix D.11}$$

In order to solve equation (Appendix D.4) by using the same power transformation:

$$z = s_2^{-2}, \qquad g(t, s_2) = H(t, z).$$

Applying the chain rule gives

$$g_t = H_t,$$

$$g_{s_2} = \frac{\partial H}{\partial z}\frac{\partial z}{\partial s_2} = -2s_2^{-3}H_z,$$

and

$$g_{s_2 s_2} = 6s_2^{-4}H_z + 4s_2^{-6}H_{zz}.$$

Substituting these expressions into equation (Appendix D.4) yields the transformed equation. Motivated by the structure of the resulting equation, we seek a solution of the form

$$H(t, z) = L(t) + M(t)z,$$

where $L(t)$ and $M(t)$ are deterministic functions to be determined. where $L(T) = M(T) = 0$.
And

$$L(t) = \frac{1}{2}\frac{\left[(\phi_2\left(\alpha_2 + (1-\lambda_2)\delta_2 - \ln s_2(t)\right) - r_0\right]^2}{\sigma_2^2}(T-t). \tag{Appendix D.12}$$

$$M(t) = 0. \tag{Appendix D.13}$$

Hence we have

$$G(T, x_1, x_2, s_1, s_2) = \frac{1}{\beta_1\beta_2}\exp\left(-\beta_1 x_1 - \beta_2 x_2 - h_1(t)\right)e^{r_0(T-t)} + (J_1(t) + K(t)s_1^{-2}) + (L_1(t) + M(t)s_2^{-2}). \tag{Appendix D.14}$$

where

$$h_1(t) = \frac{\eta_1 a\beta_2}{r_0}(1 - e^{r_0(T-t)}) - \frac{1}{2r_0}\left[b_1^2\beta_1^2 + \frac{(\beta_1 - \frac{1}{2}\beta_2)(-2\eta_1 b_1^2\beta_1)}{\eta_3} + \frac{(\beta_1 - \frac{1}{2}\beta_2)(2\eta_1^2 b_1^2\beta_1)}{\eta_3^2}\right](e^{r_0(T-t)} - e^{-r_0(T-t)}). \tag{Appendix D.15}$$

and

$$J_1(t) = \frac{1}{2}\frac{\left[(\phi_1\big(\alpha_1 + (1-\lambda_1)\delta_1 - \ln s_1(t)\big) - r_0\right]^2}{\sigma_1^2}(T-t). \tag{Appendix D.16}$$

$K(t)$ is given by equation (Appendix D.11),

$$L_1(t) = \frac{1}{2}\frac{\left[(\phi_2\big(\alpha_2 + (1-\lambda_2)\delta_2 - \ln s_2(t)\big) - r_0\right]^2}{\sigma_2^2}(T-t). \tag{Appendix D.17}$$

$M(t)$ is given by equation (Appendix D.13).

(b.) For $t_0 \leq t \leq t_1$ Then;

$$q^*(t) = \bar{q}(t) = \frac{\beta_1}{\beta_1 - \beta_2} + \frac{\eta_3 a}{(\beta_1 - \beta_2)b_1^2 e^{r_0(T-t)}} \qquad t_0 \leq t \leq t_1. \tag{Appendix D.18}$$

By substituting (3.5), (3.6), (Appendix D.18) into (2.14) and separating equations, we are going to have the following decomposed equations with separating variables:

$$h_t - r_0 h(t) + a\beta_1(\eta_1 - \eta_3) - \frac{1}{2}b_1^2\beta_1^2 e^{r_0(T-t)} = 0. \tag{Appendix D.19}$$

$$\begin{aligned} &f_t + r_0 s_1(t) f_{s_1} + \frac{1}{2}\sigma_1^2 s_1^2(t) f_{s_1 s_1} \\ &- \frac{1}{2}\frac{\left[(\phi_1\big(\alpha_1 + (1-\lambda_1)\delta_1 - \ln s_1(t)\big) - r_0\right]^2}{\sigma_1^2} = 0. \end{aligned} \tag{Appendix D.20}$$

$$\begin{aligned} &g_t + r_0 s_2(t) g_{s_2} + \frac{1}{2}\sigma_2^2 s_2^2(t) g_{s_2 s_2} \\ &- \frac{1}{2}\frac{\left[(\phi_2\big(\alpha_2 + (1-\lambda_2)\delta_2 - \ln s_2(t)\big) - r_0\right]^2}{\sigma_2^2} = 0. \end{aligned} \tag{Appendix D.21}$$

When $h(T) = 0$, the solution to equation (Appendix D.19) is;

$$h(t) = \frac{a\beta_1(\eta_1 - \eta_3)}{r_0}(1 - e^{r_0(T-t)}) - \frac{1}{4r_0}b_1^2\beta_1^2(e^{r_0(T-t)} - e^{-r_0(T-t)}). \tag{Appendix D.22}$$

Equation (Appendix D.20) is the same as (Appendix D.3) and (Appendix D.21) is the same as (Appendix D.4) and is solved by power transformation. Hence, we have

$$\begin{aligned} &G(T, x_1, x_2, s_1, s_2) = \\ &\frac{1}{\beta_1\beta_2}\exp\left(-\beta_1 x_1 - \beta_2 x_2 - h_2(t)\right)e^{r_0(T-t)} + (J_1(t) + K(t)s_1^{-2}) + (L_1(t) + M(t)s_2^{-2}). \end{aligned} \tag{Appendix D.23}$$

where

$$h_2(t) = \frac{a\beta_1(\eta_1 - \eta_3)}{r_0}(1 - e^{r_0(T-t)}) - \frac{1}{4r_0}b_1^2\beta_1^2(e^{r_0(T-t)} - e^{-r_0(T-t)}). \tag{Appendix D.24}$$

and $J_1(t)$, $K(t)$, $L_1(t)$ and $M(t)$ are given by equations (Appendix D.16), (Appendix D.11), (Appendix D.17) and (Appendix D.13).

(c.) For $0 \leq t \leq t_0$, we have $\bar{q} \leq 0$ Then;

$$q^*(t) = \bar{q}(t) = 0 \qquad 0 \leq t \leq t_0. \tag{Appendix D.25}$$

By substituting (3.5), (3.6), (Appendix D.18) into (2.14) and separating equations, we are going to have the following decomposed equations with separating variables:

$$h_t - r_0 h(t) + \beta_1\eta_1 + \frac{1}{2}b_1^2\beta_1^2 e^{r_0(T-t)} = 0. \tag{Appendix D.26}$$

$$\begin{aligned} &f_t + r_0 s_1(t) f_{s_1} + \frac{1}{2}\sigma_1^2 s_1^2(t) f_{s_1 s_1} \\ &- \frac{1}{2}\frac{\left[(\phi_1\big(\alpha_1 + (1-\lambda_1)\delta_1 - \ln s_1(t)\big) - r_0\right]^2}{\sigma_1^2} = 0 \end{aligned} \tag{Appendix D.27}$$

$$\begin{aligned} &g_t + r_0 s_2(t) g_{s_2} + \frac{1}{2}\sigma_2^2 s_2^2(t) g_{s_2 s_2} \\ &- \frac{1}{2}\frac{\left[(\phi_2\big(\alpha_2 + (1-\lambda_2)\delta_2 - \ln s_2(t)\big) - r_0\right]^2}{\sigma_2^2} = 0. \end{aligned} \tag{Appendix D.28}$$

When $h(T) = 0$, the solution to equation (Appendix D.26) is;

$$h(t) = \frac{\beta_1\eta_1}{r_0}(1 - e^{r_0(T-t)}) - \frac{1}{4r_0}b_1^2\beta_1^2(e^{r_0(T-t)} - e^{-r_0(T-t)}). \tag{Appendix D.29}$$

Equation (Appendix D.27) is the same as (Appendix D.3) and (Appendix D.28) is the same as (Appendix D.4) and is solved by power transformation. Hence, we have

$$G(T, x_1, x_2, s_1, s_2) = \frac{1}{\beta_1\beta_2}\exp\left(-\beta_1 x_1 - \beta_2 x_2 - h_3(t)\right)e^{r_0(T-t)} + (J_1(t) + K(t)s_1^{-2}) + (L_1(t) + M(t)s_2^{-2}). \tag{Appendix D.30}$$

where

$$h_3(t) = \frac{a\beta_1\eta_1}{r_0}(1 - e^{r_0(T-t)}) - \frac{1}{4r_0}b_1^2\beta_1^2(e^{r_0(T-t)} - e^{-r_0(T-t)}). \tag{Appendix D.31}$$

and $J_1(t)$, $K(t)$, $L_1(t)$ and $M(t)$ are given by equations (Appendix D.16), (Appendix D.11), (Appendix D.17) and (Appendix D.13).

Case 2. $1 < \frac{a\eta_3^2}{b_1^2\beta_2\eta_1 + b_1^2\beta_1(\eta_3-\eta_1)}$ *and* $\frac{a\eta_3^2}{b_1^2\beta_2\eta_1 + b_1^2\beta_1(\eta_3-\eta_1)} \leq e^{rt} \leq \frac{a\eta_3}{b_1^2\beta_1}$.

(a.) For $t_1 \leq t \leq T$, then $\bar{q}(t) \geq \frac{\eta_1}{\eta_3}$.

Then;

$$q^*(t) = \frac{\eta_1}{\eta_3} \qquad t_1 \leq t \leq T. \qquad \text{(Appendix D.32)}$$

Hence we have

$$G(T, x_1, x_2, s_1, s_2) = \frac{1}{\beta_1\beta_2} \exp\left(-\beta_1 x_1 - \beta_2 x_2 - h_4(t)\right) e^{r_0(T-t)} + (J_2(t) + K_1(t) s_1^{-2}) + (L_2(t) + M_1(t) s_2^{-2}). \qquad \text{(Appendix D.33)}$$

and $h_4(t) = h_1(t)$ the value is in equation (Appendix D.15), $J_2(t) = J_1(t)$, $K_1(t) = K(t)$, $L_2(t) = L_1(t)$ and $M_1(t) = M(t)$ are given by equations (Appendix D.16), (Appendix D.11), (Appendix D.17) and (Appendix D.13).

(b.) For $0 \leq t \leq t_1$, then $\bar{q} \in \big(0, \frac{\eta_1}{\eta_3}\big)$.

Then;

$$q^*(t) = \bar{q}(t) = \frac{\beta_1}{\beta_1 - \beta_2} + \frac{\eta_3 a}{(\beta_1 - \beta_2) b_1^2 e^{r_0(T-t)}} \qquad 0 \leq t \leq t_1. \qquad \text{(Appendix D.34)}$$

$$G(T, x_1, x_2, s_1, s_2) = \frac{1}{\beta_1\beta_2} \exp\left(-\beta_1 x_1 - \beta_2 x_2 - h_5(t)\right) e^{r_0(T-t)} + (J_2(t) + K_1(t) s_1^{-2}) + (L_2(t) + M_1(t) s_2^{-2}). \qquad \text{(Appendix D.35)}$$

and $h_5(t) = h_2(t)$ the value is in equation (Appendix D.24), $J_2(t) = J_1(t)$, $K_1(t) = K(t)$, $L_2(t) = L_1(t)$ and $M_1(t) = M(t)$ are given by equations (Appendix D.16), (Appendix D.11), (Appendix D.17) and (Appendix D.13).

Case 3. $1 < \frac{a\eta_3^2}{b_1^2\beta_2\eta_1 + b_1^2\beta_1(\eta_3-\eta_1)}$ *and* $\frac{a\eta_3^2}{b_1^2\beta_2\eta_1 + b_1^2\beta_1(\eta_3-\eta_1)} > e^{rt}$.

For $0 \leq t \leq T$, then $\bar{q} \geq \frac{\eta_1}{\eta_3}$.
Then;

$$q^*(t) = \frac{\eta_1}{\eta_3} \qquad 0 \leq t \leq T. \qquad \text{(Appendix D.36)}$$

Hence we have

$$G(T, x_1, x_2, s_1, s_2) = \frac{1}{\beta_1\beta_2} \exp\left(-\beta_1 x_1 - \beta_2 x_2 - h_6(t)\right) e^{r_0(T-t)} + (J_3(t) + K_2(t) s_1^{-2}) + (L_3(t) + M_2(t) s_2^{-2}). \qquad \text{(Appendix D.37)}$$

and $h_6(t) = h_1(t)$ the value is in equation (Appendix D.15), $J_3(t) = J_1(t)$, $K_2(t) = K(t)$, $L_3(t) = L_1(t)$ and $M_2(t) = M(t)$ are given by equations (Appendix D.16), (Appendix D.11), (Appendix D.17) and (Appendix D.13).

Case 4. $0 < \frac{a\eta_3^2}{b_1^2\beta_2\eta_1 + b_1^2\beta_1(\eta_3-\eta_1)} \leq 1,\ 1 < \frac{a\eta_3}{b_1^2\beta_1} < e^{r_0T}$ *and* $\frac{a\eta_3}{b_1^2\beta_1} < e^{r_0T}$.

(a.) For $t_0 < t \leq T$ then $\bar{q} \in \big(0, \frac{\eta_1}{\eta_3}\big)$.

Then;

$$q^*(t) = \bar{q}(t) = \frac{\beta_1}{\beta_1-\beta_2} + \frac{\eta_3 a}{(\beta_1-\beta_2)b_1^2e^{r_0(T-t)}} \qquad t_0 < t \leq T. \quad \text{(Appendix D.38)}$$

$G(T, x_1, x_2, s_1, s_2) =$

$$\frac{1}{\beta_1\beta_2}\exp\left(-\beta_1x_1 - \beta_2x_2 - h_7(t)\right)e^{r_0(T-t)} + (J_4(t) + K_3(t)s_1^{-2}) + (L_4(t) + M_3(t)s_2^{-2}). \quad \text{(Appendix D.39)}$$

and $h_7(t) = h_2(t)$ the value is in equation (Appendix D.24), $J_4(t) = J_1(t)$, $K_3(t) = K(t)$, $L_4(t) = L_1(t)$ and $M_3(t) = M(t)$ are given by equations (Appendix D.16), (Appendix D.11), (Appendix D.17) and (Appendix D.13).

(b.) For $0 \leq t \leq t_0$ then $\bar{q}(t) \leq 0$.

Then;

$$q^*(t) = \bar{q}(t) = 0 \qquad 0 \leq t \leq t_0. \qquad \text{(Appendix D.40)}$$

$G(T, x_1, x_2, s_1, s_2) =$

$$\frac{1}{\beta_1\beta_2}\exp\left(-\beta_1x_1 - \beta_2x_2 - h_8(t)\right)e^{r_0(T-t)} + (J_4(t) + K_3(t)s_1^{-2}) + (L_4(t) + M_3(t)s_2^{-2}). \quad \text{(Appendix D.41)}$$

and $h_8(t) = h_3(t)$ the value is in equation (Appendix D.31), $J_4(t) = J_1(t)$, $K_3(t) = K(t)$, $L_4(t) = L_1(t)$ and $M_3(t) = M(t)$ are given by equations (Appendix D.16), (Appendix D.11), (Appendix D.17) and (Appendix D.13).

Case 5. $0 < \frac{a\eta_3^2}{b_1^2\beta_2\eta_1 + b_1^2\beta_1(\eta_3-\eta_1)} \leq 1,\ 1 < \frac{a\eta_3}{b_1^2\beta_1} < e^{r_0T}$ *and* $\frac{a\eta_3}{b_1^2\beta_1} \geq e^{rt}$.

For $0 \leq t \leq T$ then $\bar{q}(t) \in \big(0, \frac{\eta_1}{\eta_3}\big)$.
Then;

$$q^*(t) = \bar{q}(t) = \frac{\beta_1}{\beta_1-\beta_2} + \frac{\eta_3 a}{(\beta_1-\beta_2)b_1^2e^{r_0(T-t)}} \qquad 0 \leq t \leq T. \quad \text{(Appendix D.42)}$$

$G(T, x_1, x_2, s_1, s_2) =$

$$\frac{1}{\beta_1\beta_2}\exp\left(-\beta_1x_1 - \beta_2x_2 - h_9(t)\right)e^{r_0(T-t)} + (J_5(t) + K_4(t)s_1^{-2}) + (L_5(t) + M_4(t)s_2^{-2}). \quad \text{(Appendix D.43)}$$

and $h_9(t) = h_2(t)$ the value is in equation (Appendix D.24), $J_5(t) = J_1(t)$, $K_4(t) = K(t)$, $L_5(t) = L_1(t)$ and $M_4(t) = M(t)$ are given by equations (Appendix D.16), (Appendix D.11), (Appendix D.17) and (Appendix D.13).

Case 6. $0 < \frac{a\eta_3^2}{b_1^2\beta_2\eta_1 + b_1^2\beta_1(\eta_3-\eta_1)} \leq 1$ *and* $0 < \frac{a\eta_3}{b_1^2\beta_1} \leq 1$.

For $0 \leq t \leq T$ then $\bar{q} \leq 0$. Then;

$$q^*(t) = \bar{q}(t) = 0 \qquad 0 \leq t \leq T. \tag{Appendix D.44}$$

$$G(T, x_1, x_2, s_1, s_2) =$$
$$\frac{1}{\beta_1\beta_2} \exp\left(-\beta_1 x_1 - \beta_2 x_2 - h_{10}(t)\right) e^{r_0(T-t)} + (J_6(t) + K_5(t)s_1^{-2}) + (L_6(t) + M_5(t)s_2^{-2}). \tag{Appendix D.45}$$

and $h_{10}(t) = h_3(t)$ the value is in equation (Appendix D.31), $J_6(t) = J_1(t)$, $K_5(t) = K(t)$, $L_6(t) = L_1(t)$ and $M_5(t) = M(t)$ are given by equations (Appendix D.16), (Appendix D.11), (Appendix D.17) and (Appendix D.13).

Appendix D.0.1 Verification of the Assumptions of Theorem 2

The candidate value functions derived in the preceding cases belong to

$$C^{1,2,2,2,2}\big([0, T] \times \mathbb{R}^2 \times \mathbb{R}_+^2\big),$$

since the functions $h(t)$, $J(t)$, $K(t)$, $L(t)$, and $M(t)$ are continuously differentiable and the exponential form of the value function is twice continuously differentiable with respect to all state variables.

Furthermore, the candidate value functions satisfy the HJB equation (Appendix B.1) together with the terminal condition

$$V(T, x_1, x_2, s_1, s_2) = U(x_1, x_2).$$

Moreover, the controls (π_1^*, π_2^*, q^*) attain the supremum in the HJB equation and satisfy the admissibility conditions stated in Definitions 2.1 and 2.2. Therefore, by the Verification Theorem, the candidate value functions coincide with the optimal value functions, and the corresponding controls are optimal.

Appendix D.0.2 Admissibility of the Optimal Controls

The optimal controls (π_1^*, π_2^*, q^*) are progressively measurable functions of the state variables. Furthermore, they satisfy the admissibility and integrability conditions stated in Definitions 2.1 and 2.2. In particular,

$$\mathbb{E}\left[\int_0^T \Big((\pi_1^*(t))^2 + (\pi_2^*(t))^2 + (q^*(t))^2\Big)\, dt\right] < \infty.$$

Moreover, the optimal reinsurance strategy satisfies

$$q^*(t) \in \left[0, \frac{\eta_1}{\eta_3}\right],$$

for all admissible parameter regimes considered in Theorem 1. Hence, the derived controls belong to the admissible control set and are therefore admissible strategies.

Appendix D.0.3 Verification of Terminal Conditions

The auxiliary functions are derived subject to the terminal conditions

$$h(T) = 0, \qquad J(T) = K(T) = 0, \qquad L(T) = M(T) = 0.$$

Substituting $(t = T)$ into the candidate value function yields

$$G(T, x_1, x_2, s_1, s_2) = \frac{1}{\beta_1\beta_2} e^{-\beta_1 x_1 - \beta_2 x_2},$$

which coincides with the prescribed terminal utility function. Therefore, the terminal boundary condition is satisfied.

Appendix D.0.4 Existence and Uniqueness

The controlled wealth and asset-price processes are governed by stochastic differential equations whose coefficients satisfy the standard local Lipschitz and linear growth conditions. Therefore, by classical existence and uniqueness results for stochastic differential equations, the state process

$$(X_1(t), X_2(t), S_1(t), S_2(t))$$

admits a unique strong solution on the interval $[0, T]$.

Furthermore, the auxiliary functions $h(t)$, $J(t)$, $K(t)$, $L(t)$, and $M(t)$ satisfy linear ordinary differential equations subject to terminal conditions. By standard ODE theory, these equations admit unique solutions on $[0, T]$. Consequently, the candidate value function is uniquely determined in each admissible regime.

Appendix E Derivation of the threshold times t_0 and t

Appendix E.1 To determine the threshold time t_0

Consider the critical equation

$$e^{r_0(T-t)} = \frac{a\eta_3^2}{b_1^2\beta_2\eta_1 + b_1^2\beta_1(\eta_3 - \eta_1)}.$$

Taking the natural logarithm on both sides yields

$$\ln\left(e^{r_0(T-t)}\right) = \ln\left(\frac{a\eta_3^2}{b_1^2\beta_2\eta_1 + b_1^2\beta_1(\eta_3 - \eta_1)}\right).$$

Using the identity $\ln(e^x) = x$, we obtain

$$r_0(T - t) = \ln\left(\frac{a\eta_3^2}{b_1^2\beta_2\eta_1 + b_1^2\beta_1(\eta_3 - \eta_1)}\right).$$

Dividing both sides by r_0 gives

$$T - t = \frac{1}{r_0}\ln\left(\frac{a\eta_3^2}{b_1^2\beta_2\eta_1 + b_1^2\beta_1(\eta_3 - \eta_1)}\right).$$

Subtracting T from both sides yields

$$-t = \frac{1}{r_0}\ln\left(\frac{a\eta_3^2}{b_1^2\beta_2\eta_1 + b_1^2\beta_1(\eta_3 - \eta_1)}\right) - T.$$

Multiplying through by -1, we obtain

$$t = T - \frac{1}{r_0}\ln\left(\frac{a\eta_3^2}{b_1^2\beta_2\eta_1 + b_1^2\beta_1(\eta_3 - \eta_1)}\right).$$

Since this value corresponds to the critical switching time between admissible control regimes, we denote it by t_0. Hence,

$$t_0 = T - \frac{1}{r_0}\ln\left(\frac{a\eta_3^2}{b_1^2\beta_2\eta_1 + b_1^2\beta_1(\eta_3 - \eta_1)}\right).$$

Appendix E.2 To determine the threshold time t_1

Consider the critical equation

$$e^{r_0(T-t)} = \frac{a\eta_3}{b_1^2\beta_1}.$$

Taking the natural logarithm on both sides yields

$$\ln\left(e^{r_0(T-t)}\right) = \ln\left(\frac{a\eta_3}{b_1^2\beta_1}\right).$$

Using the identity $\ln(e^x) = x$, we obtain

$$r_0(T-t) = \ln\left(\frac{a\eta_3}{b_1^2\beta_1}\right).$$

Dividing both sides by r_0 gives

$$T - t = \frac{1}{r_0}\ln\left(\frac{a\eta_3}{b_1^2\beta_1}\right).$$

Subtracting T from both sides yields

$$-t = \frac{1}{r_0}\ln\left(\frac{a\eta_3}{b_1^2\beta_1}\right) - T.$$

Multiplying both sides by -1, we obtain

$$t = T - \frac{1}{r_0}\ln\left(\frac{a\eta_3}{b_1^2\beta_1}\right).$$

Since this value corresponds to the critical switching time between admissible control regions, we denote it by t_1. Therefore,

$$t_1 = T - \frac{1}{r_0} \ln\left(\frac{a\eta_3}{b_1^2 \beta_1} \right).$$