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# Optimal Investment and Insurer–Reinsurer Strategies under a Geometric Mean-Reverting Model with Taxation and Dividend Effects

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## Abstract

Insurance and reinsurance companies play a vital role in modern financial markets worldwide, including developing insurance markets such as Tanzania. Most studies look at optimal investment and reinsurance choices from the insurer's point of view. However, these strategies might not serve the reinsurer's interests. This study fills that gap by exploring investment and reinsurance strategies that consider both parties' interests using a Geometric Mean-Reverting (GMR) model with dividend payments and federal income tax. Both parties invest in a risk-free asset and in-dependent risky assets, whose prices follow mean-reverting patterns. We formulate the problem as a stochastic control problem. The goal is to maximize the expected product of exponential terminal wealth utilities for both parties. By applying Hamilton–Jacobi–Bellman equations, we derive explicit optimal strategies for investment and reinsurance. The results show that the optimal reinsurance strategy is independent of the financial market parameters associated with the risky assets, including the mean-reversion, volatility, dividend, and tax parameters. Instead, it is determined by insurance and reinsurance characteristics such as safety-loadings, claim-risk parameters, and risk-aversion preferences. In contrast, the optimal investment strategies differ between the insurer and the reinsurer because of their distinct risk preferences and financial structures. Taxation significantly affects investment behavior, with lower tax rates encouraging greater investment in risky assets and consequently increasing expected returns and dividend income. Numerical experiments further illustrate the sensitivity of the optimal strategies to key model parameters. These findings emphasize the need to consider joint decision-making, mean-reverting dynamics, taxation, and dividend policies when modeling insurance and reinsurance strategies. They offer a practical approach for optimizing financial decisions in today's insurance markets.

**Keywords:** Geometric mean reversion (GMR), optimal investment, Reinsurance, Federal income tax, Dividend yield, Hamilton–Jacobi–Bellman equation

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# 1. Introduction

Investment and reinsurance are among the most important financial mechanisms employed by insurance companies to manage risk, improve solvency, and enhance long-term financial performance. Consequently, the problem of jointly determining optimal investment and reinsurance strategies has attracted considerable attention in actuarial science, insurance mathematics, and financial engineering. Recent research has increasingly focused on integrated investment-reinsurance frameworks that account for market uncertainty, stochastic factors, and the complex interactions between underwriting and financial risks. For ex-ample, (Colaneri et al., 2021) investigated optimal investment and proportional reinsurance strategies in a regime-switching market under forward preferences, while (Li et al., 2022) examined optimal investment and proportional reinsurance decisions under a mean-reverting Ornstein Uhlenbeck process. Similarly, (Zhang & Zhao, 2022) studied optimal investment–reinsurance policies in the presence of defaultable securities, and (Zhang & Li, 2023) analyzed portfolio optimization under HARA utility preferences. More recently, (Ceci & Colaneri, 2025) explored portfolio and reinsurance optimization under an unknown market price of risk, highlighting the importance of information uncertainty in insurance decision-making. Moreover, recent advances in asset-liability management and dynamic portfolio allocation have stressed the necessity of integrated risk-management frameworks that jointly consider underwriting risk, investment opportunities, solvency requirements, and capital allocation decisions (Consigli et al., 2026; Huber, 2024). Collectively, these studies demonstrate that optimal investment and reinsurance decisions remain fundamental to modern insurance operations and continue to evolve in response to increasingly complex financial, economic, and regulatory environments.

Dynamic reinsurance strategies have attracted increasing attention because insurers continuously adjust their risk exposures in response to changing market conditions, claim experience, and investment opportunities. Unlike traditional static arrangements, dynamic reinsurance frameworks enable the joint determination of investment and reinsurance decisions within a stochastic optimization setting. The concept of jointly considering the interests of insurers and reinsurers can be traced back to the pioneering work of Borch (1960), who recognized that a reinsurance arrangement that is optimal for one party may not necessarily be optimal for the other. Building on this foundation, recent studies have developed integrated insurer-reinsurer optimization models that account for the objectives of both parties while balancing risk transfer, capital preservation, and profitability. For example, (Huang & Yang, 2018) developed a robust optimization framework for maximizing the expected product of the utilities of the insurer and the reinsurer, while (Peng et al., 2020) investigated optimal investment and reinsurance decisions under model uncertainty. More recently, (Zhang & Zhao, 2022) incorporated defaultable bonds into a joint insurer-reinsurer optimization framework, and (Zhang & Li, 2023) examined optimal investment and reinsurance decisions under HARA utility preferences. Moreover, (Ceci & Colaneri, 2025) highlighted the importance of information uncertainty in port-folio and reinsurance optimization, whereas (Li et al., 2022) demonstrated the influence of mean-reverting market dynamics on optimal investment and reinsurance policies. Collectively, these studies indicate that dynamic reinsurance plays a crucial role in modern insurance risk management and that jointly considering the objectives of insurers and reinsurers leads to more realistic and economically meaningful decision-making frameworks.

Taxation and dividend policies play an important role in the financial performance of insurance and reinsurance companies because they directly influence after-tax returns, capital accumulation, and surplus management. In insurance markets, taxable income is generally derived from both underwriting and in-vestment activities, making taxation an important consideration in financial decision-making. Recent research has increasingly emphasized the incorporation of tax effects and dividend policies into investment optimization models, particularly in environments characterized by economic uncertainty and evolving regulatory conditions (Almeida et al., 2020; Fjeldstad, 2020). Dividend payments also serve as an important mechanism through which firms distribute earnings, signal financial strength, and enhance shareholder confidence, thereby influencing both investment and risk-management decisions (Almeida et al., 2020). Consequently, the interaction among taxation, dividend payments, and investment decisions has become a major research topic in modern actuarial science and financial mathematics.

At the same time, the modeling of asset-price dynamics remains a fundamental component of optimal investment and reinsurance problems. Traditional stochastic volatility models, including the Constant Elasticity of Variance (CEV) model (Cox, 1975), the Stein–Stein model (Stein & Stein, 1991), and the Heston model (Heston, 1993), have been widely employed in financial applications because they capture important market characteristics such as volatility clustering, heteroscedasticity, and leverage effects. However, these models primarily describe volatility dynamics and do not explicitly account for the tendency of asset prices to revert toward a long-run equilibrium.

In contrast, mean-reverting asset-price models explicitly incorporate the corrective mechanisms that drive asset prices back toward their fundamental values following temporary deviations. Recent research has emphasized the importance of mean reversion in long-term portfolio management, asset allocation, and risk management (Agliardi et al., 2024). Among the available mean-reverting frame-works, the Geometric Mean-Reversion (GMR) model possesses several attractive features. First, unlike arithmetic mean-reverting models, the GMR model pre-serves the positivity of asset prices. Second, it captures long-run equilibrium behavior while maintaining analytical tractability, thereby facilitating the derivation of explicit optimal investment and reinsurance strategies using stochastic control techniques. Third, the GMR model naturally reflects market-adjustment mechanisms that are particularly relevant to long-term decision-making in insurance and reinsurance markets, where asset prices often fluctuate around economically justified equilibrium levels.

Compared with stochastic volatility models such as the Heston and Stein-Stein models, the GMR framework places greater emphasis on equilibrium restoration than on volatility fluctuations. While stochastic volatility models are particularly suitable for short-term option pricing and volatility forecasting, the GMR framework is more appropriate for long-term investment and reinsurance problems in which convergence toward equilibrium plays a significant role. Despite the growing literature on optimal investment and reinsurance under various stochastic environments, limited attention has been given to jointly modeling insurer-reinsurer interactions within a geometric mean-reverting framework that simultaneously incorporates dividend payments and taxation effects. This gap motivates the present study.

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Stochastic control theory provides a powerful framework for solving optimal investment and reinsurance problems under uncertainty. In particular, dynamic programming principles and Hamilton-Jacobi-Bellman (HJB) equations have been widely employed to characterize optimal investment and reinsurance strategies in insurance and finance. Recent research continues to apply stochastic control techniques to derive optimal portfolio and reinsurance strategies un-der various sources of uncertainty, including stochastic interest rates, partial information, and mean-reverting market dynamics (Ceci & Colaneri, 2025; Li et al., 2022). The flexibility of stochastic control methods allows the incorporation of multiple interacting sources of risk while facilitating the derivation of economically meaningful optimal strategies. Consequently, stochastic control remains one of the principal analytical tools in modern actuarial science and insurance mathematics.

Motivated by the aforementioned research gap, the present study investigates the optimal investment and reinsurance decisions of an insurer and a reinsurer in a market where risky asset prices follow a geometric mean-reverting process and dividend yields and federal income taxation are explicitly incorporated. Unlike the study of (Mbigili, 2012), which considered a mean-reverting portfolio optimization problem for a single investor, the present study jointly models the insurer and reinsurer within a unified optimization framework, thereby providing a more realistic representation of insurer-reinsurer interactions. Moreover, the framework developed in (Mwigilwa et al., 2025) is extended by incorporating strategic insurer–reinsurer interactions together with dividend payments and taxation effects. Specifically, both parties are allowed to invest independently in risk-free and risky assets whose prices follow a geometric mean-reverting process with dividend and taxation features, while the optimization objective is formulated as the maximization of the expected product of their utility functions. Using dynamic programming principles, the corresponding Hamilton-Jacobi-Bellman (HJB) equations are derived and solved in closed form. Numerical simulations are conducted to examine the effects of key model parameters and provide economic insights into the roles of mean reversion, taxation, dividend policies, risk aversion, and reinsurance arrangements in optimal decision-making.

The remainder of the paper is organized as follows. Section 2 presents the model formulation.

Section 3 presents the model solution. Section 4 presents the analytical results of the model. Numerical simulations and discussion of the results are provided in Section 5. Conclusions and recommendations are presented in Section 6. Detailed derivations and supplementary results are pro-vided in Appendices Appendix A, Appendix B, Appendix C, Appendix D and Appendix E

# 2. Model formulation

Let $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions, where $\mathbb{F} = \{\mathbb{F} : 0 \leq t \leq T\}$ is a filtration representing the information available to the investor and insurer over time. We assume that $\mathscr{F} = \mathbb{F}$_T at the terminal time T, where $T > 0$ is a fixed time horizon for the investment and reinsurance problem.

More precisely, the filtration $\mathbb{F}$ describes the evolution of all relevant information available to the decision-maker from the initial time $t = 0$ up to the terminal time $t = T$. Thus, $\mathbb{F}$ represents the information available at time t, while $\mathbb{F}$_T contains all information available at the end of the investment and reinsurance period.

Throughout the analysis, unless otherwise stated, all stochastic processes are assumed to be adapted to the filtration $\mathbb{F} = \{\mathbb{F} : 0 \leq t \leq T\}$. That is, for every stochastic process $X = (X\ ) \leq \leq T$ considered in the model, the value X is measurable with respect to $\mathbb{F}$ for each $t \in [0,T]$. This ensures that the processes and decisions considered in the investment and reinsurance strategies depend only on information available up to the corresponding time t.

## 2.1 Surplus process

The aggregate claim process of the insurer is assumed to follow the diffusion approximation

$$dC(t) = adt - b_1 dW^0(t), \tag{2.1}$$

where $(a > 0)$ denotes the average claim rate, $(b1 > 0)$ is the claim volatility parameter, and $(W^0(t))$ is a standard Brownian motion.

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The insurer calculates premiums according to the expected value principle. Therefore, the premium rate is given by

$$c_1 = (1 + \eta_1)a, \tag{2.2}$$

where ($\eta_1 > 0$) is the insurer's safety loading coefficient. The safety loading represents the additional premium charged above the expected claim cost in order to cover administrative expenses, uncertainty, and profit requirements. Consequently, the insurer's surplus process before reinsurance is obtained as

$$dR(t) = c_1 dt - dC(t). \tag{2.3}$$

Substituting equations 2.1 and 2.2, into 2.3 we obtain

$$\begin{aligned} dR(t) &= (1 + \eta_1)adt - \ adt - b_1 dW^0(t) \\ &= a\eta_1 dt + b_1 dW^0(t). \end{aligned}$$

Suppose that the insurer purchases proportional reinsurance. Let $q(t) \in [0, 1]$ denote the proportion of claim risk transferred to the reinsurer. Consequently, the insurer retains the proportion $1 - q(t)$ of the aggregate claim risk. The reinsurance contract is priced according to the expected value principle with reinsurer safety loading coefficient $\eta_3 > 0$. Following standard assumptions in the actuarial literature, we assume that $\eta_3 > \eta_1$, reflecting the fact that reinsurance protection is generally more expensive than direct insurance.

Under this proportional reinsurance arrangement, the insurer pays an additional loading proportional to the ceded risk $q(t)$, while the claim volatility is reduced according to the retained proportion $1 - q(t)$. Hence, the surplus process of the insurer is given by

$$dR_1^q(t) = a(\eta_1 - \eta_3 q(t))dt + b_1(1 - q(t))dW^0(t),$$

while the surplus process of the reinsurer is given by

$$dR_2^q(t) = a\eta_3 q(t)dt + b_1 q(t)dW^0(t).$$

These dynamics imply that a larger reinsurance proportion q(t) reduces the insurer's exposure to claim risk while increasing the reinsurer's exposure. At the same time, the insurer incurs a higher reinsurance cost through the loading term a $\eta_3$q(t), which represents the compensation paid to the reinsurer for assuming the additional risk.

### 2.2 The Financial Market

The dynamics of the risk-free asset $S_0 = \{S_0(t), 0 \leq t \leq T\}$ for both the insurer and the reinsurer at time t are described by

$$\begin{cases} dS_0(t) = r_0 S_0(t)\,dt, \\ S_0(0) = s_0. \end{cases} \tag{2.4}$$

For the insurer, the price process of the risky asset $S_1 = \{S_1(t), 0 \leq t \leq T\}$ under the geometric mean reversion (GMR) model is given by

$$\begin{cases} dS_1(t) = \phi_1\big(\alpha_1 - \ln S_1(t)\big)S_1(t)\,dt + \sigma_1 S_1(t)\,dW^1(t), \\ S_1(0) = \bar{s}_1. \end{cases} \tag{2.5}$$

Similarly, for the reinsurer, the price process of the risky asset $S^2 = \{S^2(t), 0 \leq t \leq T\}$ under the GMR model is given by

$$\begin{cases} dS_2(t) = \phi_2\big(\alpha_2 - \ln S_2(t)\big)S_2(t)\,dt + \sigma_2 S_2(t)\,dW^2(t), \\ S_2(0) = \bar{s}_2. \end{cases} \tag{2.6}$$

The parameters $\phi^1$ and $\phi^2$ represent the speeds of mean reversion, while $\alpha^1$ and $\alpha^2$ denote the long-term equilibrium levels, i.e., the values toward which the asset prices tend to revert over time. The parameters $\sigma 1$ and $\sigma^2$ represent the volatilities of the risky assets. The processes $W^1(t)$ and $W^2(t)$ are standard Brownian motions, assumed to be independent of each other and also independent of $W^0(t)$.

The assumption that the Brownian motions $(W^0)$, $(W^1)$, and $(W^2)$ are mutually independent is adopted primarily to simplify the model and facilitate the derivation of explicit solutions. Although this assumption is commonly employed in the stochastic control literature, it may not fully capture the dependence structure between underwriting risk and financial market risk observed in practice, particularly during periods of economic stress. Allowing the Brownian motions to be correlated would introduce additional covariance terms into the wealth dynamics and the Hamilton–Jacobi–Bellman (HJB) equation, thereby increasing the mathematical complexity of the problem. Although the general solution methodology would remain applicable, the resulting optimal controls would require a separate analysis. Extending the proposed framework to incorporate correlated sources of risk constitutes a natural direction for future research.

It is further assumed that the risky assets held by the insurer and the rein-surer pay continuous dividends at constant yields $(\delta_1)$ and $(\delta_2)$, respectively. In the absence of taxation, dividend payments provide an additional source of return to investors. However, when dividend income is subject to taxation, only a proportion $(1 - \lambda_i)$ of the dividend yield is retained, where $\lambda_i \in (0, 1)$ de-notes the income tax rate applicable to investor i, (i = 1, 2). Consequently, the effective after-tax dividend yield is given by $(1 - \lambda_i)\delta_i$.

To incorporate the effects of dividends and taxation within the geometric mean-reversion framework, we adopt the modeling assumption that the after-tax dividend yield shifts the effective long-run equilibrium level of the risky asset. Accordingly, the long-run mean parameter $(\alpha_i)$ is adjusted to

$$\alpha_i + (1 - \lambda_i)\delta_i, \quad i = 1, 2.$$

This specification reflects the fact that dividend payments increase the long-term attractiveness and expected growth potential of the asset, while taxation reduces the net benefit received by the investor. In a standard asset-pricing framework, dividends are typically introduced as an additive component of the drift term. In contrast, the present formulation incorporates after-tax dividend payments through the long-run mean parameter to preserve the geometric mean-reversion structure of the asset-price dynamics. Accordingly, the original processes in (2.5) and (2.6) are modified as follows.

Therefore if the insurer is subject to income tax on dividends, the dynamics of the risky asset $S_1(t)$ become

$$\begin{cases} dS_1(t) = \phi_1\Big(\alpha_1 + (1-\lambda_1)\delta_1 - \ln S_1(t)\Big)S_1(t)\,dt + \sigma_1 S_1(t)\,dW^1(t), \\ S_1(0) = \bar{s}_1, \end{cases} \tag{2.7}$$

where $\lambda_1$ is the income tax rate coefficient of the insurer.

Similarly, if the reinsurer is also subject to income tax on dividends, the dynamics of the risky asset $S_2(t)$ become

$$\begin{cases} dS_2(t) = \phi_2\Big(\alpha_2 + (1-\lambda_2)\delta_2 - \ln S_2(t)\Big)S_2(t)\,dt + \sigma_2 S_2(t)\,dW^2(t), \\ S_2(0) = \bar{s}_2, \end{cases} \qquad ((2.8)$$

where $\lambda_2$ is the income tax rate coefficient of the reinsurer.

For the insurer, let $\pi_1(t)$ denote the amount invested in the risky asset at time t. Consequently, the amount invested in the risk-free asset is $X_1(t) - \pi_1(t)$. A pair $\kappa_1 = (\pi_1(t), q(t))$ represents an investment-reinsurance strategy for the insurer.

The corresponding wealth process of the insurer is given by

$$\begin{aligned} dX_1^{\kappa_1}(t) &= \pi_1(t)\frac{dS_1(t)}{S_1(t)} + \big(X_1^{\kappa_1}(t) - \pi_1(t)\big)\frac{dS_0(t)}{S_0(t)} + dR_1^q(t) \\ &= \Big[r_0 X_1^{\kappa_1}(t) + \pi_1(t)\big(\phi_1(\alpha_1 + (1-\lambda_1)\delta_1 - \ln S_1(t)) - r_0\big) + \\ &\quad a_1(\eta_1 - \eta_3 q(t))\Big]dt \quad + \pi_1(t)\sigma_1\,dW^1(t) + b_1\big(1-q(t)\big)\,dW^0(t). \end{aligned} \tag{2.9}$$

For the reinsurer, let $\pi_2$(t) denote the amount invested in the risky asset at time t. Consequently, the amount invested in the risk-free asset is $X_2$(t) − $\pi_2$ (t). A pair $\kappa_2$ = ($\pi_2$(t), q(t)) represents an investment-reinsurance strategy for the reinsurer.

Then, the wealth process for the reinsurer is given by

$$
\begin{aligned}
dX_2^{\kappa_2}(t) &= \pi_2(t)\frac{dS_2(t)}{S_2(t)} + \left(X_2^{\kappa_2}(t) - \pi_2(t)\right)\frac{dS_0(t)}{S_0(t)} + dR_2^q(t) \\
&= \Big[r_0 X_2^{\kappa_2}(t) + \pi_2(t)\big(\phi_2(\alpha_2 + (1-\lambda_2)\delta_2 - \ln S_2(t)) - r_0\big) + a_1\eta_3 q(t)\Big]dt \\
&\quad + \pi_2(t)\sigma_2\, dW^2(t) + b_1 q(t)\, dW^0(t). \qquad (2.10)
\end{aligned}
$$

**Definition 2.1. (Admissible strategies for the insurer)**

Let $\pi_1$(t) denote the amount invested by the insurer in the risky asset $S_1$(t) at time t. A strategy $\kappa_1$ = {($\pi_1$(t), q(t)), t ∈ [0, T ]} is said to be admissible if:

1. $\{\kappa_1(t)\}_{t\geq 0}$ is $\{\mathcal{F}_t\}_{t\in[0,T]}$-progressively measurable;
2. $\kappa_1 \in \Pi_1$, where

$$
\Pi_1 = \left\{\kappa_1 : \mathbb{E}\left[\int_t^T \left(q^2(t) + \pi_1^2(t)\right) dt\right] < \infty, \quad q(t) \in [0,1]\right\},
$$

and $\Pi_1$ is called the admissible set of $\kappa_1$.

**Definition 2.2. (Admissible strategies for the reinsurer)**

Let $\pi_2$(t) denote the amount invested by the reinsurer in the risky asset $S_2$(t) at time t. A strategy

$\kappa_2 = \{(\pi_2(t), q(t)), t \in [0, T]\}$ is said to be admissible if:

1. $\{\kappa_2(t)\}_{t\geq 0}$ is $\{\mathcal{F}_t\}_{t\in[0,T]}$-progressively measurable;
2. $\kappa_2 \in \Pi_2$, where

$$\Pi_2 = \left\{ \kappa_2 : \mathbb{E}\left[\int_t^T \left(q^2(t) + \pi_2^2(t)\right) dt\right] < \infty, \quad q(t) \in [0,1] \right\},$$

and $\Pi_2$ is called the admissible set of $\kappa_2$.

The exponential utility functions of the insurer and the reinsurer are given by

$$U_1(x_1) = -\frac{1}{\beta_1} e^{-\beta_1 x_1}, \qquad U_2(x_2) = -\frac{1}{\beta_2} e^{-\beta_2 x_2},$$

where $\beta_1 > 0$ and $\beta_2 > 0$ are the risk aversion coefficients.

The product of the insurer's and reinsurer's utility functions is considered the following:

$$U(x_1, x_2) = U_1(x_1)U_2(x_2) = \frac{1}{\beta_1\beta_2} e^{-\beta_1 x_1 - \beta_2 x_2}. \tag{2.11}$$

The objective function is based on the product of the insurer's and reinsurer's exponential utility functions. This formulation provides a cooperative framework in which the preferences of both parties are incorporated into a single optimisation criterion. The product utility defined in Equation (2.11) preserves the risk-aversion characteristics of both participants while maintaining analytical tractability. Although alternative approaches, such as Nash bargaining, Pareto optimisation, and Stackelberg game formulations, may also be employed to model insurer-reinsurer interactions, the multiplicative utility criterion pro-vides a convenient representation of cooperative decision-making and has been widely adopted in the literature. The investigation of alternative cooperative and game-theoretic objective functions is left for future research..

The objective of the insurer and reinsurer is to maximize the expected utility of their terminal wealth:

$$\max_{(\pi_1(t),\,\pi_2(t),\,q(t))} \mathbb{E}\big[U\big(X_1(T), X_2(T)\big)\big].$$

The value function of the problem is defined as

$$G(t, x_1, x_2, s_1, s_2) = \sup_{(\pi_1(t),\,\pi_2(t),\,q(t))} \mathbb{E}\Big[U\big(X_1(T), X_2(T)\big)\,\Big|\, S_1(t) = s_1,\ S_2(t) = s_2,\ X_1(t) = x_1,\ X_2(t) = x_2\Big]. \tag{2.12}$$

We first offer the following notations;

$$C^{1,2,2}([0,T], \mathbb{R}, \mathbb{R}, \mathbb{R}^+, \mathbb{R}^+) = \{G(t, x_1, x_2, s_1, s_2) | G(t, \cdot, \cdot, \cdot, \cdot, \cdot) \text{ is once continuously differentiable on } [0,T] \text{ and } G(t, x_1, x_2) \text{ is twice continuously differentiable on } \mathbb{R} \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+\Big\}.$$

## 2.3 Hamilton-Jacobi-Bellman Equation

Applying Itô's formula to the value function $G(t, x_1, x_2, s_1, s_2)$, together with the controlled wealth dynamics of the insurer and the reinsurer and the corresponding risky-asset dynamics, yields the infinitesimal generator A$\kappa_1$ , $\kappa_2$ . For completeness, the detailed derivation of the infinitesimal generator and the corre-sponding Hamilton-Jacobi-Bellman (HJB) equation is provided in Appendix A. By the dynamic programming principle, the value function satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\sup_{\kappa_1, \kappa_2} \{\mathcal{A}^{\kappa_1, \kappa_2} G\} = 0. \tag{2.13}$$

Hence, for any sufficiently smooth function ($G(t, x_1, x_2, s_1, s_2)$), the generator A$\kappa_1$ , $\kappa_2$ takes the form:

$$
\begin{aligned}
\mathcal{A}^{\kappa_1,\kappa_2}G =& G_t + \Big[r_0 x_1 + \pi_1(t)\Big(\phi_1\big(\alpha_1 + (1-\lambda_1)\delta_1 - \ln s_1\big) - r_0\Big) + a\big(\eta_1 - \eta_3 q(t)\big)\Big]G_{x_1} \\
&+ \Big[r_0 x_2 + \pi_2(t)\Big(\phi_2\big(\alpha_2 + (1-\lambda_2)\delta_2 - \ln s_2\big) - r_0\Big) + a\eta_3 q(t)\Big]G_{x_2} \\
&+ \Big[\phi_1\big(\alpha_1 + (1-\lambda_1)\delta_1 - \ln s_1\big)s_1\Big]G_{s_1} + \Big[\phi_2\big(\alpha_2 + (1-\lambda_2)\delta_2 - \ln s_2\big)s_2\Big]G_{s_2} \\
&+ \frac{1}{2}\Big[\pi_1^2(t)\sigma_1^2 + b_1^2(1-q(t))^2\Big]G_{x_1x_1} + \frac{1}{2}\Big[\pi_2^2(t)\sigma_2^2 + b_1^2 q^2(t)\Big]G_{x_2x_2} \\
&+ \frac{1}{2}\sigma_1^2 s_1^2 G_{s_1s_1} + \frac{1}{2}\sigma_2^2 s_2^2 G_{s_2s_2} + \pi_1(t)\sigma_1^2 s_1, G_{x_1s_1} + \pi_2(t)\sigma_2^2 s_2, G_{x_2s_2} + \\
& b_1^2 q(t)\big(1-q(t)\big)G_{x_1x_2}.
\end{aligned} \tag{2.14}
$$

with $G(T, x_1, x_2, s_1, s_2) = U(x_1, x_2)$.

To rigorously establish the optimality of the obtained solution, it is essential to demonstrate that the candidate value function represents the true value function of the stochastic control problem. Additionally, it must be shown that the corresponding controls are admissible and effectively maximize the Hamilton-Jacobi-Bellman (HJB) equation. The verification theorem presented below out-lines sufficient conditions for these properties, thereby confirming the optimality of the proposed strategy.

**Theorem 1** (Verification Theorem)**.** *Suppose that there exists a function*

$$
V \in C^{1,2,2,2,2}\big([0,T] \times \mathbb{R}^2 \times \mathbb{R}_+^2\big)
$$

*satisfying the Hamilton–Jacobi–Bellman equation*

$$
\sup_{(\pi_1,\pi_2,q)\in\mathcal{A}} \{V_t + \mathcal{L}^{\pi_1,\pi_2,q}V\} = 0, \tag{2.15}
$$

*with terminal condition*

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$$V(T, x_1, x_2, s_1, s_2) = U(x_1, x_2).$$

*Assume further that*

1. *For every admissible control*

$$(\pi_1, \pi_2, q) \in \mathcal{A},$$

*the controlled state process*

$$(X_1, X_2, S_1, S_2)$$

*admits a unique strong solution.*

2. *The function* $(V)$ *satisfies the growth condition*

$$\mathbb{E}\left[\sup_{0\le t\le T} |V(t, X_1(t), X_2(t), S_1(t), S_2(t))|\right] < \infty.$$

3. *The local martingale arising from Itô's formula is a true martingale.*

4. *There exists an admissible control*

$$(\pi_1^*, \pi_2^*, q^*) \in \mathcal{A}$$

*such that*

$$(\pi_1^*, \pi_2^*, q^*) = \arg \max_{(\pi_1,\pi_2,q)\in\mathcal{A}} \left(V_t + \mathcal{L}^{\pi_1,\pi_2,q} V\right).$$

5. *The control*

$$(\pi_1^*, \pi_2^*, q^*)$$

*satisfies the admissibility condition,*

$$\mathbb{E}\int_0^T \left(|\pi_1^*(t)|^2 + |\pi_2^*(t)|^2 + |q^*(t)|^2\right) dt < \infty.$$

*Then*

$$V(t, x_1, x_2, s_1, s_2) = G(t, x_1, x_2, s_1, s_2),$$

*and*

$$(\pi_1^*, \pi_2^*, q^*)$$

is an optimal control.

**[Proof. Appendix B]**

## 2.4 Model solution

In this study, the analysis is restricted to the parameter regime ($\beta_2 > \beta_1$), where the reinsurer is assumed to be more risk-averse than the insurer. This assumption makes it possible to analytically analyze the optimization issue and provides explicit formulations for the optimal controls. While this model is relevant to explore the effect of a moderately conservative reinsurer on the collaborative decision-making process, it does not cover all practical situations. In particular, large, well-diversified reinsurers may have lower effective risk aversion than insurers, which corresponds to the scenario ($\beta_1 \geq \beta_2$). The consideration of this alternative parameter regime may lead to distinct optimal techniques and deserves separate investigation. Consequently, an attractive direction to ex-tend the present model is to include the case ($\beta_1 \geq \beta_2$), which is an intriguing direction for future research.

The optimal investment strategy's first-order condition of optimality $\pi_1(t)$ is;

$$\pi_1^*(t) = \frac{-\left[(\phi_1\big(\alpha_1 + (1-\lambda_1)\delta_1 - \ln s_1(t)\big) - r_0\right]G_{x_1} - \sigma_1^2 s_1(t) G_{x_1 s_1}}{\sigma_1^2 G_{x_1 x_1}}. \quad (3.1)$$

The optimal investment strategy's first-order condition of optimality $\pi_2(t)$ is;

$$\pi_2(t) = \frac{-\left[(\phi_2\big(\alpha_2 + (1-\lambda_2)\delta_2 - \ln s_2(t)\big) - r_0\right]G_{x_2} - \sigma_2^2 s_2(t) G_{x_2 s_2}}{\sigma_2^2 G_{x_2 x_2}}. \quad (3.2)$$

To solve the HJB equation, we seek a candidate value function that preserves the exponential structure of the terminal utility function. Since the objective function is based on exponential utilities and the wealth variables ($x_1$) and ($x_2$) enter linearly into the state dynamics, it is natural to consider a separation-of-variables representation in which the dependence on wealth variables is exponential, while the effects of the risky asset prices are captured through auxiliary functions.

Accordingly, the following candidate value function is proposed:

$$G(t, x_1, x_2, s_1, s_2)] = \frac{1}{\beta_1 \beta_2} \exp\left[(-\beta_1 x_1 - \beta_2 x_2 - h(t))e^{r_0(T-t)} + f(t, s_1) + g(t, s_2)\right], \tag{3.3}$$

where ($h(t)$) is a deterministic function and ($f(t, s_1)$) and ($g(t, s_2)$) are sufficiently smooth functions to be determined.

The choice of this form is motivated by the terminal condition and the exponential nature of the utility functions. Substituting this representation into the HJB equation allows the state variables to separate, leading to a system of equations for ($h(t)$), ($f(t, s_1)$), and ($g(t, s_2)$).

Given $h(T) = f(T) = g(T) = 0$ as the boundary conditions.

The partial derivatives of the candidate value function are given by;

$$
\begin{aligned}
&G_t = \big[-r_0 e^{r_0(T-t)}(-\beta_1 x_1 - \beta_2 r_2 x_2 - h(t)) - h_t e^{r_0(T-t)} + f_t + g_t\big]G,\\
&G_{x_1} = -\beta_1 e^{r_0(T-t)}G, \quad G_{x_1x_1} = \beta_1^2 e^{2r_0(T-t)}G,\\
&G_{x_2} = -\beta_2 e^{r_0(T-t)}G, \quad G_{x_2x_2} = \beta_2^2 e^{2r_0(T-t)}G,\\
&G_{s_1} = f_{s_1}G, \quad G_{s_1s_1} = (f_{s_1}^2 + f_{s_1s_1})G,\\
&G_{s_2} = g_{s_2}G, \quad G_{s_2s_2} = (g_{s_2}^2 + g_{s_2s_2})G,\\
&Gx_1s_1 = -\beta_1 f_{s_1} e^{r_0(T-t)}G, \quad Gx_2s_2 = -\beta_2 g_{s_2} e^{r_0(T-t)}G,\\
&\quad Gx_1x_2 = \beta_1\beta_2 e^{2r_0(T-t)}G.
\end{aligned} \tag{3.4}
$$

Substituting the derivatives in (3.4) into the HJB equation yields

$$\sup_{\kappa_1,\kappa_2} \{\mathcal{A}^{\kappa_1,\kappa_2}G\} = 0.$$

Since the candidate value function ($G(t, x_1, x_2, s_1, s_2)$) is strictly positive for all admissible states and times, dividing the above equation by (G) gives an equivalent nonlinear equation involving only the functions (h(t)), (f (t, $s_1$)), (g(t, $s_2$)), and the control variables (q(t)), ($\pi_1$(t)), and ($\pi_2$(t)). The first-order optimality conditions

$$\frac{\partial}{\partial q}\left(\frac{\mathcal{A}^{\kappa_1,\kappa_2}G}{G}\right) = 0, \qquad \frac{\partial}{\partial \pi_1}\left(\frac{\mathcal{A}^{\kappa_1,\kappa_2}G}{G}\right) = 0, \qquad \frac{\partial}{\partial \pi_2}\left(\frac{\mathcal{A}^{\kappa_1,\kappa_2}G}{G}\right) = 0,$$

yield the candidate optimal controls $(q^*(t)), (\pi^*(t))$ and $(\pi^*(t))$.

Furthermore, the second-order conditions confirm that the obtained controls maximize the Hamiltonian. Substituting the optimal controls back into the reduced HJB equation produces a system of differential equations satisfied by (h(t)), (f (t, $s_1$)), and (g(t, $s_2$)). Solving this system subject to the terminal conditions

$$h(T) = 0, \qquad f(T, s_1) = 0, \qquad g(T, s_2) = 0,$$

shows that the proposed value function satisfies the HJB equation and there-fore verifies the candidate solution.

By substituting (3.4) into (3.1) we obtain the following optimal strategy π∗(t):

$$\pi_1^*(t) = \frac{\left[(\phi_1\big(\alpha_1 + (1-\lambda_1)\delta_1 - \ln s_1(t)\big) - r_0\right] + \sigma_1^2 s_1(t) f_{s_1}}{\sigma_1^2 \beta_1 e^{r_0(T-t)}}. \qquad (3.5)$$

Similarly, by substituting (3.4) into (3.2) we obtain the following optimal strategy $\pi^*(t)$:

$$\pi_2^*(t) = \frac{\left[(\phi_2\big(\alpha_2 + (1-\lambda_2)\delta_2 - \ln s_2(t)\big) - r_0\right] + \sigma_2^2 s_2(t) g_{s_2}}{\sigma_2^2 \beta_2 e^{r_0(T-t)}}. \qquad (3.6)$$

Differentiating the HJB equation with respect to the reinsurance control q(t) and setting the result equal to zero yields the first-order condition

$$q^*(t) = \frac{\eta_3 a G_{x_1} - \eta_3 a G_{x_2} + b_1^2 G_{x_1 x_1} - b_1^2 G_{x_1 x_2}}{b_1^2 G_{x_1 x_1} + b_1^2 G_{x_2 x_2} - 2 b_1^2 G_{x_1 x_2}}. \qquad (3.7)$$

Substituting the derivatives given in (3.4) into the above expression and simplifying, we obtain

$$q^*(t) = \frac{\beta_1}{\beta_1 - \beta_2} + \frac{\eta_3 a}{(\beta_1 - \beta_2) b_1^2 e^{r_0(T-t)}}. \qquad (3.8)$$

It is worth noting that Equation (3.8) represents the unconstrained interior solution. When $\beta_2 > \beta_1$, the denominator $\beta_1 - \beta_2$ becomes negative, implying that the unconstrained optimal retention level may also become negative. Since the admissible retention strategy satisfies $0 \leq q(t) \leq 1$, a negative retention level has no economic interpretation and is therefore infeasible. In such situations, the interior solution is no longer admissible, and the optimal strategy is obtained from the boundary of the admissible control set. Consequently, the optimal policy switches to the full-reinsurance regime, namely $q^*(t) = 0$, as established in Theorem 2. Economically, this reflects the fact that when the reinsurer is substantially more risk-averse than the insurer, the unconstrained optimization problem ceases to admit a feasible interior solution, and the optimal contract is determined by the admissibility constraints.

## 3. Results

**Theorem 2.** *Define*

$$t_0 = T - \frac{1}{r_0} \ln \left( \frac{a\eta_3^2}{b_1^2\beta_2\eta_1 + b_1^2\beta_1(\eta_3 - \eta_1)} \right),$$

*and*

$$t_1 = T - \frac{1}{r_0} \ln \left( \frac{a\eta_3}{b_1^2\beta_1} \right).$$

*The threshold times ($t_0$) and ($t_1$) arise from the critical values at which the candidate optimal controls switch between different admissible regions. Specifically, they are obtained by solving the equations*

$$e^{r_0(T-t)} = \frac{a\eta_3^2}{b_1^2\beta_2\eta_1 + b_1^2\beta_1(\eta_3 - \eta_1)}$$

*and*

$$e^{r_0(T-t)} = \frac{a\eta_3}{b_1^2\beta_1},$$

*respectively, for the time variable (t). The proof of derivations have been pro-vided in Appendix E. These threshold values partition the planning horizon into intervals corresponding to different optimal reinsurance regimes.*

*For the problem of maximizing the expected product of the utility functions of the insurer and the reinsurer under the geometric mean reversion model with dividend payments and federal income taxes, the optimal investment strategies are given by*

$$\begin{cases} \pi_1^*(t) = \dfrac{\left[(\phi_1\left(\alpha_1+(1-\lambda_1)\delta_1-\ln s_1(t)\right)-r_0\right]}{\sigma_1^2\beta_1 e^{r_0(T-t)}}, \\ \pi_2^*(t) = \dfrac{\left[(\phi_2\left(\alpha_2+(1-\lambda_2)\delta_2-\ln s_2(t)\right)-r_0\right]}{\sigma_2^2\beta_2 e^{r_0(T-t)}}. \end{cases} \tag{4.1}$$

The corresponding optimal reinsurance strategy ($q^*(t)$) and value function are classified according to the parameter regions summarized in Table 4 on Appendix D. The resulting solutions can be grouped into three economically distinct regimes, namely constant retention, full reinsurance, and dynamic retention, as illustrated in Figure 1.

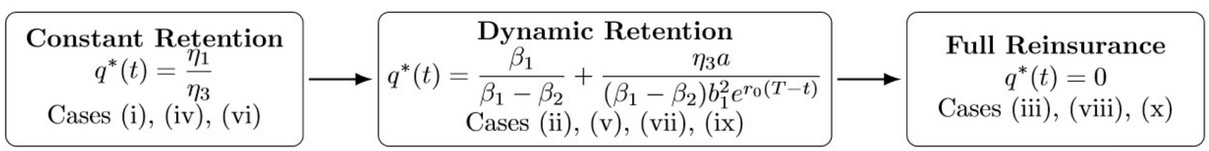


**Figure 1:** Taxonomy of optimal reinsurance regimes.

**Remark 1.** *The optimal retention strategy can be classified into three economically distinct regimes: constant retention, dynamic retention, and full reinsurance. These regimes reflect different balances between insurance risk, reinsurance cost, investment opportunities, and risk preferences.*

***(i)*** ***Constant retention regime***

*When*

$$q^*(t) = \frac{\eta_1}{\eta_3},$$

*the insurer retains a constant proportion of the claim risk throughout the investment horizon. This retention level is determined by the relative safety loadings of the insurer and the reinsurer and represents a stable risk-sharing arrangement. Economically, the insurer retains risk up to the point where the marginal benefit of retaining additional premium income equals the marginal cost* *associated with higher claim exposure.*

*It is important to note that the geometric mean-reversion parameters, dividend yields, and taxation coefficients do not appear explicitly in this expression. This occurs because the optimal reinsurance decision in this regime is governed primarily by the insurance market structure and the pricing of reinsurance. Nevertheless, the financial market characteristics continue to influence the optimal investment strategies ((π∗, π∗)) and the associated value function. Thus, while the retention level remains constant, the overall wealth optimization problem still reflects the effects of mean reversion, dividends, and taxation.*

***(ii) Dynamic retention regime***

*When*

$$q^*(t) = \frac{\beta_1}{\beta_1 - \beta_2} + \frac{\eta_3 a}{(\beta_1 - \beta_2) b_1^2 e^{r_0(T-t)}},$$

*the insurer adjusts its retention level continuously over time. This regime arises because the insurer must simultaneously balance claim risk, reinsurance cost, and future investment opportunities. Unlike the constant-retention case, the optimal strategy depends explicitly on the remaining time to maturity through the factor ($e^{r_0(T-t)}$ )). Consequently, the insurer's willingness to retain risk changes as the terminal horizon approaches.*

*Economically, this regime reflects active risk management. The insurer dynamically modifies the amount of risk transferred to the reinsurer in response to changes in the value of future wealth and the opportunity cost of reinsurance. Since the optimal investment strategies are directly affected by the geometric mean-reversion dynamics, dividend yields, and taxation, these financial-market features indirectly influence the dynamic retention decision through the underlying optimization framework. Therefore, this regime provides the strongest evidence of the interaction between insurance and investment decisions.*

***(iii) Full reinsurance regime***

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*When*

$$q^*(t) = 0,$$

*the insurer transfers all claim risk to the reinsurer. In this situation, the cost of retaining risk outweighs the potential benefits of receiving additional premium income. As a result, complete risk transfer becomes optimal.*

*Economically, this regime corresponds to highly risk-averse behavior or market conditions under which insurance risk is relatively expensive to bear. By eliminating underwriting risk, the insurer protects its surplus and avoids exposure to claim volatility. The insurer's performance is then driven primarily by investment decisions rather than underwriting activities. Although the retention level itself does not depend explicitly on the mean-reversion, dividend, or taxation parameters, these factors remain important because they affect the investment opportunities available to both the insurer and the reinsurer and therefore influence the resulting value function.*

*Overall, the results demonstrate that the financial-market characteristics introduced in this study primarily affect the investment component of the optimization problem and the associated value function. Their influence on the reinsurance decision depends on the regime under consideration: in some cases the retention level is determined mainly by insurance-market parameters, while in others it evolves dynamically through its interaction with the investment opportunity set and the remaining planning horizon.*

***[Proof.*** **Appendix D]**

## 3.1. Economic Interpretation of the Optimal Controls

The optimal investment and reinsurance strategies are given by

$$\pi_1^*(t) = \frac{\phi_1\big(\alpha_1 + (1-\lambda_1)\delta_1 - \ln s_1(t)\big) - r_0}{\sigma_1^2 \beta_1 e^{r_0(T-t)}}. \tag{4.2}$$

$$\pi_2^*(t) = \frac{\phi_2\big(\alpha_2 + (1-\lambda_2)\delta_2 - \ln s_2(t)\big) - r_0}{\sigma_2^2 \beta_2 e^{r_0(T-t)}}. \tag{4.3}$$

$$q^*(t) = \frac{\beta_1}{\beta_1 - \beta_2} + \frac{\eta_3 a}{(\beta_1 - \beta_2) b_1^2 e^{r_0(T-t)}}. \tag{4.4}$$

The optimal investment strategies given in (4.2) and (4.3) exhibit the classical trade-off between expected return and risk. In both expressions, the numerators represent the excess expected returns generated by the geometric mean-reverting assets relative to the risk-free rate, while the denominators capture the combined effects of market volatility and risk aversion. Consequently, the optimal allocation to the risky asset increases when the expected excess return is high and decreases when either the asset volatility or the investor's degree of risk aversion increases.

Equations (4.2) and (4.3) also reflect the mean-reverting nature of the under-lying asset-price dynamics. When the current asset price falls below its long-run equilibrium level, the term

$$\alpha_i + (1 - \lambda_i)\delta_i - \ln s_i(t), \qquad i = 1, 2,$$

becomes relatively large, thereby increasing the expected return component and encouraging a larger investment in the risky asset. Conversely, when the asset price rises above its equilibrium level, the expected return decreases, leading to a smaller risky investment position. Therefore, both the insurer and the reinsurer adjust their portfolios in response to deviations in asset prices from their long-run equilibrium values.

Furthermore, the factor ($e^{r_0(T-t)}$) appearing in (4.2) and (4.3) implies that the optimal investment strategies depend on the remaining investment horizon. As the end of the period draws near, the chance to recover from bad market movements decreases, prompting the insurer and reinsurer to modify their exposure to risky assets accordingly.

The optimal reinsurance strategy given by (4.4) reflects the trade-off between retaining underwriting profits and transferring insurance risk to the reinsurer. The expression shows that the optimal level of risk transfer depends directly on the relative risk-aversion parameters ($\beta_1$) and ($\beta_2$). An increase in the insurer's risk-aversion coefficient ($\beta_1$) increases the demand for reinsurance and therefore raises the optimal level of risk transfer. In contrast, an increase in the reinsurer's risk-aversion coefficient ($\beta_2$) reduces the willingness of the reinsurer to accept additional risk, thereby decreasing the optimal reinsurance level. Consequently, the optimal reinsurance arrangement is determined by the interaction between insurance risk, reinsurance cost, and the risk preferences of both parties.

Overall, Equations (4.2)– (4.4) demonstrate that the optimal investment and reinsurance decisions are jointly influenced by market conditions, claim uncertainty, mean-reversion effects, the remaining investment horizon, and the risk preferences of the insurer and the reinsurer. These results highlight the importance of integrating investment and reinsurance decisions within a unified stochastic optimization framework.

## 3.2. Numerical Simulation

Numerical simulations are presented in this section to illustrate the theoretical results. Unless otherwise specified, the parameters values in Tables 1, 2 and 3 are adopted throughout the numerical analysis. The parameter are selected to provide economically meaningful numerical illustrations and facilitate the analysis of the optimal investment and reinsurance strategies. Since the primary objective of this study is to investigate the analytical properties and qualitative behaviour of the proposed model rather than calibrate it to a specific insurance market, the selected parameter values represent realistic and plausible scenarios encountered in insurance and reinsurance practices.

Table 1: Baseline parameters used in the numerical analysis

| **Parameter** | $\beta_1$ | $\beta_2$ | $T$ | $t$ | $r_0$ | $\delta_1$ |
|---|---|---|---|---|---|---|
| **Value** | **0.5** | **0.6** | **10** | **5** | **0.02** | **0.1** |
| Economic Meaning | Risk aversion of in-surer | Risk aversion of reinsurer | Investment horizon | Evaluation time | Risk-free interest rate | Dividend yield of insurer's asset |
| Source/Selection Rationale | Representative Moderate risk-aversion level used for illustration | Chosen slightly larger than $\beta 1$ | Long-term planning horizon | Intermediate evaluation point | Low-risk annual return - | Moderate dividend-paying asset |

Table 2: Baseline parameters used in the numerical analysis (continued)

| **Parameter** | $\delta_2$ | $\eta_3$ | $a$ | $b_1$ | $s_1$ | $s_2$ |
|---|---|---|---|---|---|---|

| Value | 0.4 | 2 | 2 | 1 | 10 | 12 |
|---|---|---|---|---|---|---|
| Economic Meaning | Dividend yield of reinsurer's asset | Reinsurance safety loading | Claim intensity parameter | Claim volatility | Initial insurer asset price | Initial reinsurer as-set price |
| Source/Selection Rationale | Higher dividend-yield scenario | Relatively high reinsurance loading | Representative claim intensity | Standard normalization value | Benchmark initial asset value | Heterogeneous as-set position |

Moreover, the baseline parameter values are broadly consistent with those commonly adopted in the insurance and financial mathematics literature. In particular, the values ($\delta_1$ = 0.1) and ($\delta_2$ = 0.4) represent moderate and relatively high dividend-yield environments, respectively, thereby facilitating an assessment of the effects of dividend policies on the optimal investment strategies. The parameter ($\eta_3$ = 2) represents a relatively high reinsurance safety loading and is selected to illustrate the influence of reinsurance costs on the optimal retention strategy. In addition, the planning horizon T = 10 represents a long-term investment period commonly considered in insurance applications, while t = 5 denotes an intermediate evaluation time within the investment horizon.

Table 3: Baseline parameters used in the numerical analysis (continued)

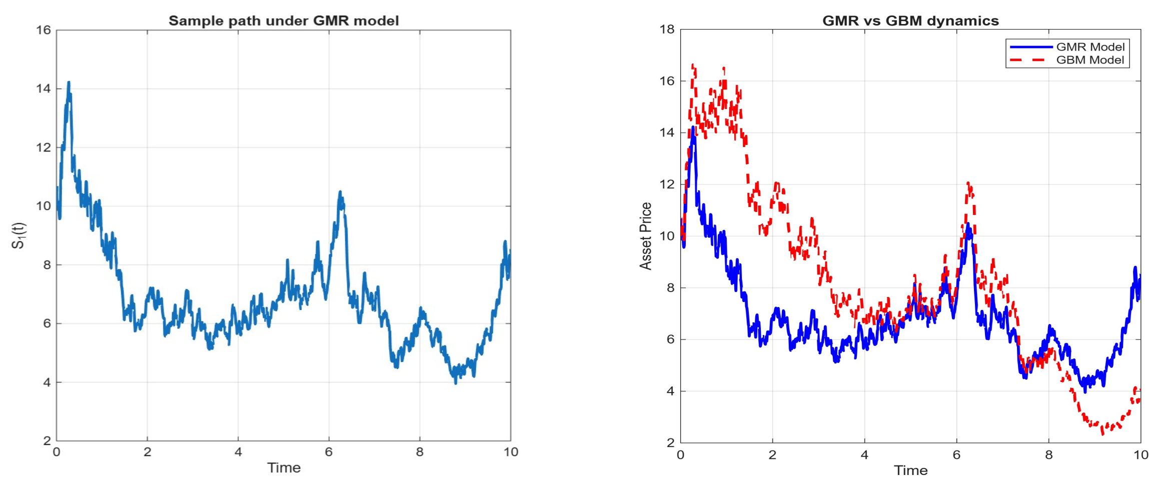
| Parameter | $\phi_1$ | $\phi_2$ | $\alpha_1$ | $\alpha_2$ | $\lambda_1$ | $\lambda_2$ | $\sigma_1$ | $\sigma_2$ | $\eta_1$ |
|---|---|---|---|---|---|---|---|---|---|
| Value | 1.2 | 1.1 | 2 | 2.5 | 0.1 | 0.4 | 0.3 | 0.35 | 1 |
| Economic Meaning | Mean-reversion speed of insurer's asset | Mean-reversion Speed of reinsurer's asset | Long-run equilibrium level of insurer's asset | Long-run equilibrium level of reinsurer's asset | Insurer's income tax rate | Reinsurer's income tax rate | Volatility of insurer's risky asset | Volatility of reinsurer's risky asset | Insurer's safety loading coefficient - |
| Source/Selection Rationale | Moderate mean reversion | Moderate mean reversion | Representative long-run level | Higher long- run equilibrium level | Lower tax- rate scenario | Higher tax-rate scenario | Moderate market volatility | Higher market volatility | Representative insurer safety loading |

Since several model parameters are difficult to estimate accurately from publicly available data, particularly in the absence of calibration to a specific insurance market, the numerical experiments are designed to illustrate the sensitivity of the optimal strategies to changes in key parameters. This approach facilitates an assessment of the robustness of the main qualitative findings and offers economic perspectives on the effects of taxation, dividend policies, risk aversion, and reinsurance arrangements on optimal decision-making.

The simulations in Figure 2 were generated using

$$\phi_1 = 0.5, \quad \alpha_1 = 2, \quad \sigma_1 = 0.3, \quad S_1(0) = 10,$$

together with the baseline parameters reported in Table 1 and Table 2. Figure 2(a) illustrates a representative sample path of the risky asset price ($S_1(t)$) under the geometric mean-reversion (GMR) model. Although the asset price experiences stochastic fluctuations arising from market shocks, it exhibits a tendency to revert toward its long-run equilibrium level. This behavior is generated by the mean-reverting drift component, which moderates persistent deviations from equilibrium.




**a) Sample path under the GMR model** **b) Comparison between GMR and GBM models**

**Figure 2:** Illustration of asset price dynamics. (a) shows a simulated sample path under the geometric mean reversion (GMR) model, where the asset price fluctuates around a long-term equilibrium level. (b) compares the GMR and geometric Brownian motion (GBM) models, highlighting that the GBM model does not exhibit mean-reverting behavior and instead follows a non-stationary trend.

Figure 2(b) compares the asset-price dynamics under the GMR and geo-metric Brownian motion (GBM) models. Unlike the GBM model, which lacks a mean-reverting mechanism and may exhibit prolonged departures from its long-run equilibrium, the GMR model incorporates an adjustment mechanism that moderates such deviations. Consequently, the GMR trajectory exhibits greater long-term stability than the GBM trajectory.

From an economic perspective, the GMR framework is particularly suitable for long-term investment and reinsurance decisions because it combines short-term uncertainty with convergence toward a long-run equilibrium. This characteristic makes it appropriate for modeling assets whose prices are influenced by underlying economic fundamentals. Figure 2(b) provides a qualitative comparison of the two models, while a more comprehensive statistical comparison based on Monte Carlo simulations is left for future research.

Unlike stochastic volatility models such as the CEV, Stein–Stein, and Hes-ton models, the GMR framework explicitly incorporates equilibrium adjustment while preserving analytical tractability. This feature makes the model particularly suitable for long-term investment and reinsurance problems in which deviations from equilibrium are expected to be temporary.

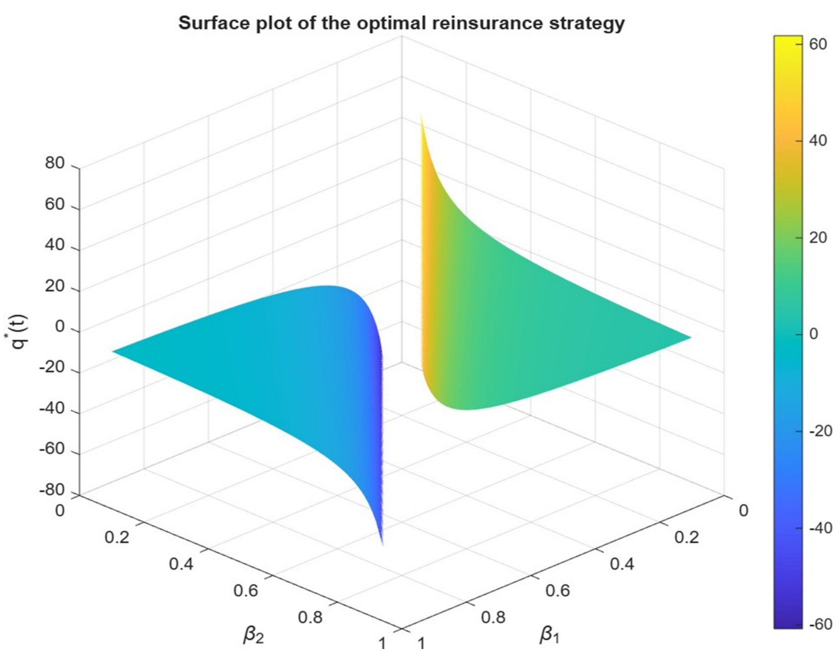



**Figure 3:** Surface plot of the optimal reinsurance strategy $q*(t)$ with respect to the risk aversion parameters $\beta_1$ (insurer) and $\beta_2$ (reinsurer). The figure illustrates the sensitivity of the optimal reinsurance decision to changes in risk preferences, showing that higher risk aversion leads to increased risk transfer.

Figure 3 illustrates the optimal reinsurance strategy $(q^*(t))$ as a function of the insurer's and reinsurer's risk-aversion coefficients, ($\beta_1$) and ($\beta_2$), based on the dynamic-retention regime

$$q^*(t) = \frac{\beta_1}{\beta_1 - \beta_2} + \frac{\eta_3 a}{(\beta_1 - \beta_2) b_1^2 e^{r_0(T-t)}},$$

derived in Theorem 2. This regime is considered because it explicitly depends on both ($\beta_1$) and ($\beta_2$), thereby illustrating how the risk preferences of the insurer and the reinsurer jointly influence the optimal reinsurance decision. In contrast, the constant-retention strategy ($q^*(t) = \eta_1/\eta_3$) and the full-reinsurance strategy ($q^*(t) = 0$) are independent of these parameters and therefore do not provide insight into the role of risk aversion.

To improve the clarity of the numerical illustration, a small neighbourhood around $\beta_1 = \beta_2$ has been excluded from the figure. At $\beta_1 = \beta_2$, the denominator ($\beta_1 - \beta_2$) in the analytical expression vanishes, causing the unconstrained interior solution to become undefined. Consequently, the vertical discontinuity observed in the original surface plot corresponds to a mathematical singularity rather than an economically meaningful phenomenon. In this situation, the first-order conditions no longer yield a feasible interior solution, and the optimal reinsurance strategy is instead determined by the boundary of the admissible control region.

The numerical results show that the optimal reinsurance strategy is highly sensitive to the relative levels of risk aversion of the insurer and the reinsurer. As the insurer's risk-aversion coefficient ($\beta_1$) increases, the optimal reinsurance level ($q^*(t)$) also increases. Since ($q^*(t)$) represents the proportion of risk transferred to the reinsurer, this indicates that a more risk-averse insurer has a stronger incentive to purchase reinsurance in order to reduce underwriting risk and stabilize future wealth.

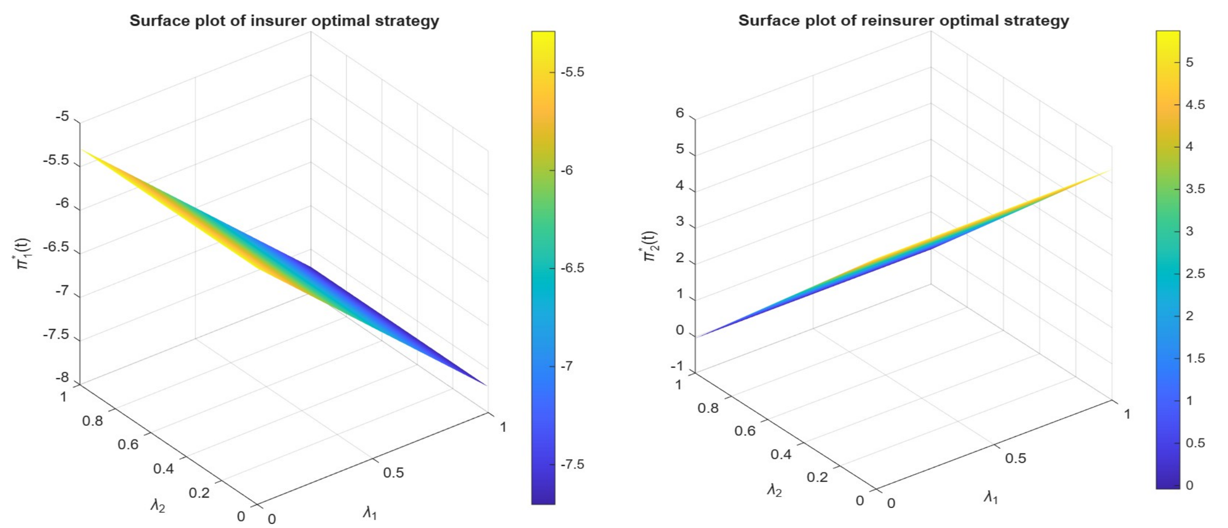
Conversely, increasing the reinsurer's risk-aversion coefficient ($\beta_2$) decreases the optimal level of risk transfer. Economically, a more risk-averse reinsurer is less willing to accept additional underwriting risk, thereby reducing the amount of risk that can be optimally ceded under the reinsurance contract. Overall, the results demonstrate that the optimal reinsurance arrangement is primarily determined by the relative risk preferences of the contracting parties. When the insurer is substantially more risk-averse than the reinsurer, greater risk transfer is optimal. In contrast, as the reinsurer becomes increasingly risk-averse, the optimal level of reinsurance declines, highlighting the importance of jointly accounting for the attitudes toward risk of both parties when designing optimal reinsurance contracts.

Figures 4(a) and 4(b) are generated from the optimal investment strategies derived in Equation (4.1), namely

$$\pi_1^*(t) = \frac{\phi_1\big(\alpha_1 + (1-\lambda_1)\delta_1 - \ln s_1(t)\big) - r_0}{\sigma_1^2 \beta_1 e^{r_0(T-t)}},$$

and

$$\pi_2^*(t) = \frac{\phi_2\big(\alpha_2 + (1-\lambda_2)\delta_2 - \ln s_2(t)\big) - r_0}{\sigma_2^2 \beta_2 e^{r_0(T-t)}}.$$




**a) Surface plot of the insurer's optimal investment strategy** **b) Surface plot of the reinsurer's optimal investment strategy**

**Figure 4:** Surface plots of optimal investment strategies. (a) shows the insurer's optimal investment strategy, $\pi^*(t)$, while (b) shows the reinsurer's optimal in-vestment strategy, $\pi^*(t)$, as functions of the tax rates $\lambda_1$ and $\lambda_2$. The results indicate that increasing taxation reduces investment in risky assets, reflecting the negative impact of taxes on expected returns.

These figures illustrate the effect of the tax parameters ($\lambda_1$) and ($\lambda_2$) on the optimal investment decisions of the insurer and the reinsurer, respectively. It follows directly from the above expressions that the tax parameters enter the investment strategies through the after-tax dividend components $((1 - \lambda_1)\delta_1)$ and $((1 - \lambda_2)\delta_2)$. Consequently, an increase in taxation reduces the effective dividend return received from the risky asset and lowers its attractiveness as an investment opportunity.

The numerical results confirm this theoretical relationship. As the tax rates increase, the optimal proportions invested in risky assets decrease for both the insurer and the reinsurer. Economically, higher taxation reduces the net re-turn generated by risky investments, encouraging both parties to shift part of their wealth toward the risk-free asset. This behavior is consistent with rational portfolio selection, where investors reduce exposure to risky assets when the expected after-tax return declines. A comparison of Figures 4(a) and 4(b) further reveals that the sensitivity of the investment strategy to taxation differs between the insurer and the reinsurer. This difference arises from variations in the underlying model parameters, including the risk-aversion coefficients $(\beta_1)$ and $(\beta_2)$, the dividend yields $(\delta_1)$ and $(\delta_2)$, and the surplus dynamics of the two parties. In particular, the reinsurer is characterized by a higher safety loading and dividend yield, which amplifies the impact of taxation on its investment decisions.

Overall, the figures demonstrate that taxation plays an important role in determining optimal portfolio allocation. Higher tax rates reduce the effective benefits of risky investments and therefore lead to more conservative investment strategies for both the insurer and the reinsurer.

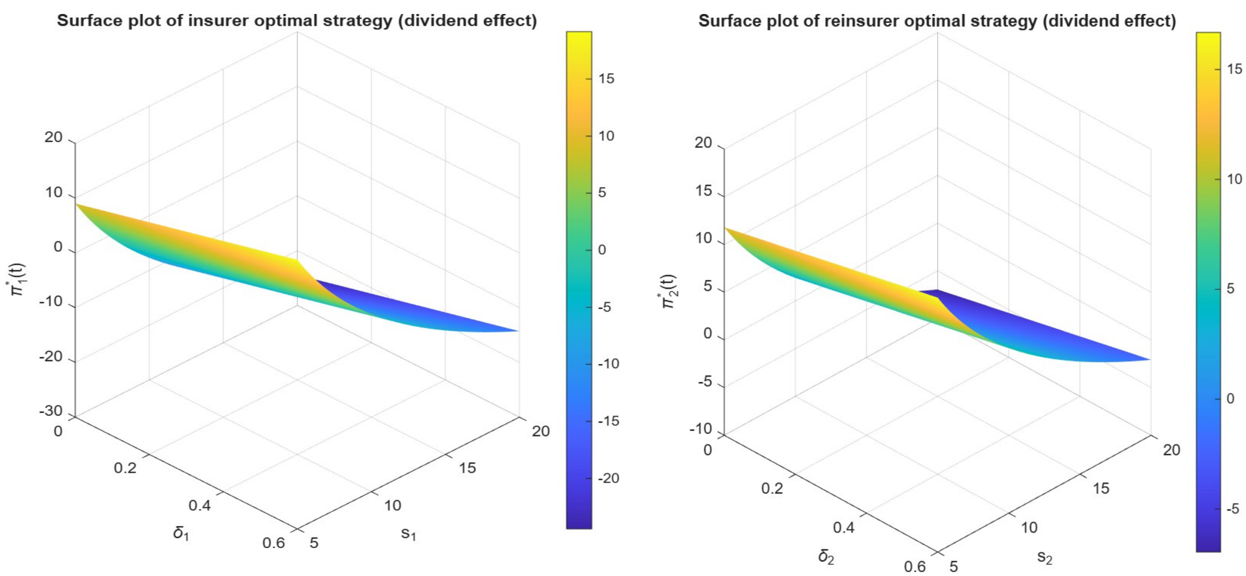
Figures 5(a) and 5(b) are generated from the optimal investment strategies given in Equation (4.1), namely,

$$\pi_1^*(t) = \frac{\phi_1\big(\alpha_1 + (1-\lambda_1)\delta_1 - \ln s_1(t)\big) - r_0}{\sigma_1^2\beta_1 e^{r_0(T-t)}},$$

and

$$\pi_2^*(t) = \frac{\phi_2\big(\alpha_2 + (1-\lambda_2)\delta_2 - \ln s_2(t)\big) - r_0}{\sigma_2^2\beta_2 e^{r_0(T-t)}}.$$

The figures illustrate the effects of the dividend yields $(\delta_1)$ and $(\delta_2)$ on the optimal investment strategies of the insurer and the reinsurer, respectively. Since the dividend yields enter the optimal controls through the terms $((1 - \lambda_1)\delta_1)$ and $((1 - \lambda_2)\delta_2)$, an increase in dividends raises the expected return of the risky assets and consequently leads to larger optimal investments. The numerical results confirm this positive relationship, indicating that dividend-paying assets become more attractive as the associated dividend yield increases. The figures also show that the optimal investment levels decrease as the asset prices ($s_1$ ) and ($s_2$) rise. This effect is driven by the logarithmic terms ($\ln s_1(t)$) and ($\ln s_2(t)$) in the optimal controls.




**a) Surface plot of the insurer's optimal investment strategy**

**b) Surface plot of the reinsurer's optimal investment strategy**

**Figure 5:** Surface plots of optimal investment strategies. (a) Represents the insurer's optimal investment strategy, $\pi^*(t)$, while (b) represents the reinsurer's optimal investment strategy, $\pi^*(t)$, as functions of the dividend yields $\delta_1$ and $\delta_2$. The results indicate that increasing dividend yields enhances the expected return of risky assets, thereby encouraging higher investment levels for both the insurer and the reinsurer.

Under the geometric mean-reversion frame-work, higher asset prices imply a weaker mean-reversion effect and a lower expected appreciation toward the long-run equilibrium level. As a result, both parties reduce their exposure to the risky assets when market prices become relatively high. A comparison of Figures 5(a) and 5(b) further suggests that the reinsurer's optimal investment strategy is more responsive to changes in dividend yield. This difference arises from the distinct parameter values characterizing the insurer and reinsurer, particularly their risk preferences, dividend structures, and surplus dynamics. Overall, the results demonstrate that dividend policy plays an important role in portfolio selection within the proposed insurance–reinsurance framework. Higher dividend yields encourage investment in risky assets, whereas elevated asset prices reduce investment demand through the mean-reversion mechanism embedded in the GMR model.

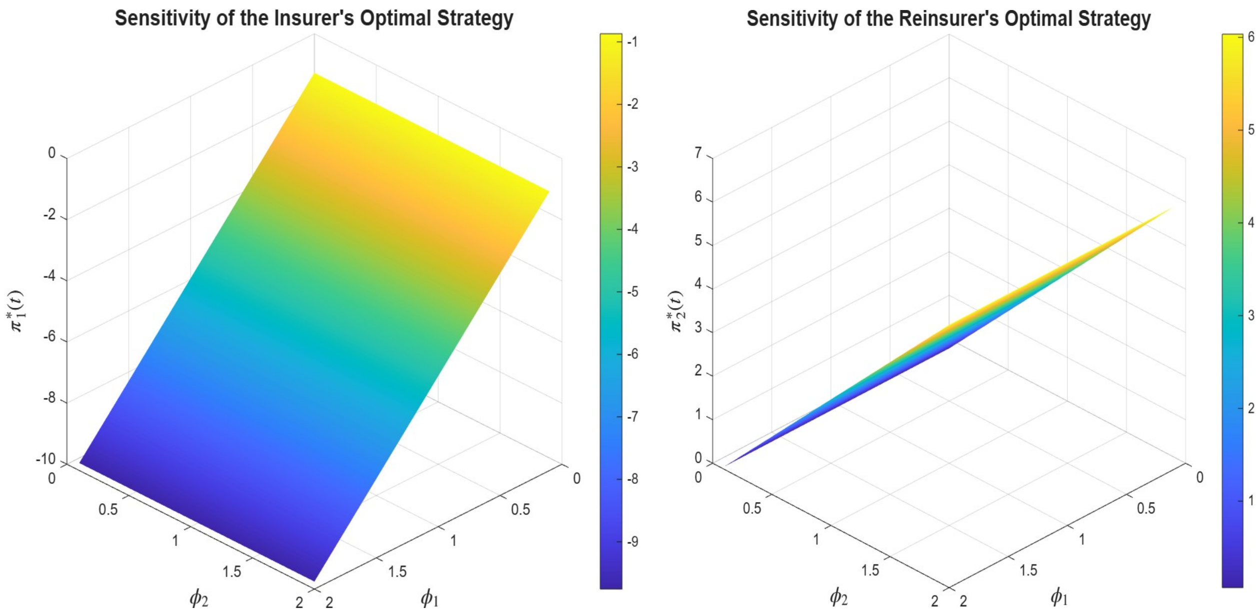
The results show that the insurer's optimal strategy is primarily influenced by its own mean-reversion speed parameter $\phi_1$, whereas the reinsurer's optimal strategy is mainly affected by $\phi_2$. These findings indicate that the speed at which asset prices revert toward their long-run equilibrium levels plays an important role in determining optimal portfolio allocations. Figures 6(a) and 6(b) are generated from the optimal investment strategies given in Equation (4.1), namely

$$\pi_1^*(t) = \frac{\phi_1\big(\alpha_1 + (1-\lambda_1)\delta_1 - \ln s_1(t)\big) - r_0}{\sigma_1^2 \beta_1 e^{r_0(T-t)}},$$

and

$$\pi_2^*(t) = \frac{\phi_2\big(\alpha_2 + (1-\lambda_2)\delta_2 - \ln s_2(t)\big) - r_0}{\sigma_2^2 \beta_2 e^{r_0(T-t)}}.$$

Figure 6(a) presents the sensitivity of the insurer's optimal investment strategy to the mean-reversion speed parameters, while Figure 6(b) shows the corresponding sensitivity for the reinsurer. The figures demonstrate that the mean-reversion characteristics of each risky asset primarily influence its corresponding optimal investment strategy. As the mean-reversion speed increases, the expected adjustment toward the long-run equilibrium level becomes stronger, which alters the attractiveness of the risky asset and consequently affects the optimal portfolio allocation. These results highlight the importance of accounting for equilibrium adjustment dynamics when formulating optimal investment strategies for insurers and reinsurers.




**a) Surface plot of the insurer's optimal investment strategy**

**b) Surface plot of the reinsurer's optimal investment strategy**

**Figure 6:** Sensitivity analysis of the optimal investment strategies with respect to the mean-reversion speed parameters $\phi_1$ and $\phi_2$. Figure (a) illustrates the insurer's optimal investment strategy, $\pi*(t)$, while Figure (b) illustrates the reinsurer's optimal investment strategy, $\pi*(t)$.

## 3.3. Practical Implications for Insurers and Reinsurers

The results of this study provide several practical insights for insurance and reinsurance companies operating in environments characterized by market uncertainty, taxation, and dividend payments. First, the optimal investment strategies indicate that both insurers and reinsurers should account for mean-reverting asset dynamics when allocating capital between risky and risk-free assets. Un-like models based solely on geometric Brownian motion, the geometric mean-reversion model captures the tendency of asset prices to move toward their long-run equilibrium levels, thereby providing additional information for port-folio allocation decisions.

Second, the optimal reinsurance strategy highlights the importance of incorporating the risk preferences of both the insurer and the reinsurer when designing reinsurance contracts. The results indicate that more risk-averse insurers tend to transfer a larger proportion of risk, whereas more risk-averse reinsurer's become less willing to accept additional risk exposure. Consequently, effective reinsurance arrangements require a balance between the insurer's demand for risk transfer and the reinsurer's willingness to provide coverage.

Third, the analysis demonstrates that taxation and dividend policies can significantly influence investment and reinsurance decisions. Since taxation directly affects after-tax returns and dividend payments influence cash flow allocation, insurance companies should account for these factors when developing long-term capital management and risk management strategies. Ignoring taxation and dividend effects may lead to suboptimal investment allocations and inefficient reinsurance decisions.

Furthermore, the joint optimization framework developed in this study pro-vides a useful decision-support tool for insurers and reinsurers seeking to co-ordinate investment and risk-transfer decisions. By simultaneously considering investment opportunities, underwriting risk, taxation, dividend policies, and risk preferences, the proposed model offers a more comprehensive approach to financial decision-making than frameworks that analyze investment and reinsurance decisions separately.

From a managerial perspective, the findings suggest that insurers and reinsurers should regularly reassess their investment and reinsurance policies in response to changes in market conditions, regulatory requirements, and corporate risk preferences. Such adjustments can improve capital efficiency, enhance solvency, and contribute to the long-term financial stability of both parties.

# 4. Conclusion and Recommendation

The problem of optimal investment and reinsurance strategies for insurers and reinsurers has attracted considerable attention in recent years, resulting in a substantial body of literature. The present study investigates this problem un-der the Geometric Mean-Reversion (GMR) model while incorporating dividend payments and federal income taxation. This framework is motivated by the significant influence of taxation policies and dividend distributions on the financial performance of insurers and reinsurers.

Specifically, an optimal investment–reinsurance problem is formulated by jointly considering the interests of both parties. The proposed model consists of one risk-free asset and two risky assets for each agent, with the risky asset prices following the GMR process in the presence of dividend payments and taxation. Using stochastic control theory, the corresponding Hamilton–Jacobi–Bellman (HJB) equation is derived, and explicit optimal investment and reinsurance strategies are obtained.

Compared with the classical geometric Brownian motion (GBM) model, the GMR model provides a more realistic representation of asset-price dynamics by incorporating mean reversion toward a long-run equilibrium. In contrast, the GBM model does not account for this equilibrium adjustment and there-fore permits persistent deviations from fundamental values. Consequently, the GMR model allows asset prices to fluctuate around their long-run equilibrium levels while preserving analytical tractability. This behavior is supported by empirical evidence from commodity, energy, and insurance markets, whereas set prices commonly exhibit mean-reverting characteristics. Figure 2 illustrates this difference by showing that asset prices under the GMR model fluctuate around a long-run equilibrium, whereas those generated by the GBM model do not exhibit mean-reverting behavior.

The numerical results further support these observations. Figure 2 demonstrates that the GMR model captures both short-term market fluctuations and long-term equilibrium behavior, thereby providing a more realistic representation of asset-price dynamics. Consequently, investment decisions based on models that ignore mean reversion may fail to capture long-term market behaviour accurately, potentially leading to suboptimal investment and reinsurance strategies. These findings highlight the importance of selecting an appropriate asset-price model when developing optimal investment and reinsurance policies.

The main findings of the study are summarized as follows:

i. The optimal reinsurance strategy is independent of the financial-market parameters of the risky assets, including the mean-reversion, volatility, dividend, and tax parameters. Instead, it depends on insurance and reinsurance characteristics such as claim-risk parameters, safety loadings, and risk-aversion preferences.

ii. The optimal investment strategies of the insurer and reinsurer differ due to their distinct risk-aversion levels, dividend structures, and tax considerations.

iii. The optimal investment strategies are primarily driven by the financial-market parameters, taxation, dividend yields, and risk-aversion preferences, while the insurance business parameters do not appear explicitly in the investment controls.
iv. Lower income tax rates encourage greater investment in risky assets by both the insurer and the reinsurer, thereby increasing expected returns and dividend income.

From the economic perspective, the graphs can also be used to draw meaningful conclusions. As illustrated by the surface graphs, higher dividend yields correspond to high-risky asset investments because they contribute positively to higher profits. On the other hand, high taxes will result in low investment in risky assets since they have negative impacts on the profitability of firms. In practice, firms usually make more investments when the after-tax returns are favorable and limit risky investments in higher tax conditions. Given the current state of the global economy, which is marked by rising inflation rates, tightened monetary policy, and market instability, there are usually price deviations for most of the asset classes from their intrinsic values. However, this tendency is usually rectified after some time. Therefore, it is possible to see that the GMR model fits well for this situation and is appropriate for use in long-term decisions in the insurance and reinsurance industries.

From a policy perspective, the findings of this study are relevant to insurance and reinsurance markets where taxation, dividend policies, and investment decisions play an important role in financial stability. The numerical results demonstrate that taxation, dividend policies, and risk preferences can significantly influence optimal investment and reinsurance decisions. In particular, lower tax rates increase the attractiveness of risky assets, whereas higher tax rates encourage more conservative investment behavior.

Although the proposed model is not calibrated using country-specific data, it provides a general theoretical framework that can be applied to both developed and developing insurance markets. In particular, the framework may provide useful insights for emerging insurance sectors, such as Tanzania, where the insurance and reinsurance industries continue to evolve. The results may therefore assist policymakers, regulators, and practitioners in understanding how taxation, dividend policies, and long-term asset dynamics can influence investment and risk-management decisions.

It should be emphasized, however, that the present study is primarily theoretical and does not seek to provide Tanzania-specific policy recommendations. The development of a fully calibrated country-specific model was beyond the scope of this work because of the limited availability of comprehensive data on insurance and reinsurance operations, dividend policies, investment portfolios, and other market-specific parameters. Future research could extend the proposed framework by calibrating it to Tanzanian market data or to other national insurance markets as suitable datasets become available.

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